On the Existence of Hermitian-Yang-Mills Connections in Stable Vector Bundles

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The Yang-Mills equations were introduced by theoretical physicists and are now accepted as a basic ingredient in particle theory. In the past decade these equations have become important in mathematics in two separate areas. It was observed early on that the twistor formalism of Penrose and the Atiyah-Singer index theory can be employed to good purpose in order to describe special solutions, called anti-selfdual solutions; see [29], [3]. Using techniques of algebraic geometry, Atiyah-Drinfeld-Hitchin-Manin [2] were able to describe all such solutions over S^4 (or $\mathbb{R}^4$) in terms of holomorphic bundles on $\mathbb{CP}^3$. More recently a simpler description using stable vector bundles was obtained by Donaldson [8]. The techniques of partial differential equations were used by Taubes [25], [26] to construct solutions on arbitrary four-manifolds. These results and compactness theorems of the first author were incorporated into a general theory of Simon Donaldson to obtain beautiful and spectacular results on the differential structures on four-manifolds; see [10]-[11]. It is an observation dating essentially back to Yang's formalism of using $\mathbb{C}^2$ instead of $\mathbb{R}^4$ that the Taubes solutions give rise to holomorphic vector bundles over Kähler surfaces; see [30].

The theory of holomorphic vector bundles is central to algebraic geometry. The concept of a stable vector bundle was introduced by Mumford in his study of moduli for bundles [18]. This theme has been pursued by several mathematicians, notably Takemoto, Horrocks, Gieseker, Maruyama and Barth. An important contribution was made by Bogomolov [4], who showed that, for a projective variety M with Pic $(M) = \mathbb{Z}$ and E a stable bundle on M, the inequality

$$\left(c_2 - \frac{k-1}{2k} c_1^2\right) \cup \omega^{n-2} \geqq 0$$

is valid. Here $k = \operatorname{rank} E$, c, and c_2 are the first and second Chern classes, and ω is a polarization for which E is stable.

Bogomolov's work inspired the work of Miyaoka [17] on the inequality $3c_2 \geqq c_1^2$ for algebraic surfaces of general type. Independently, the second author also gave a proof of this inequality at the same time. While the methods of Miyaoka and Bogomolov are essentially based on algebraic geometry, the method of the second author used the techniques of partial differential equations to

Communications on Pure and Applied Mathematics, Vol. XXXIX S257–S293 (1986)
 CCC 0010-3640/86/0S0257-37$04.00

construct a Kähler-Einstein metric in his solution of the Calabi conjecture in [31]. The Kähler-Einstein method has the advantage of giving insight into the equality case $3c_2 = c_1^2$ as well as generalizing to arbitrary dimension.

The Hermitian-Yang-Mills connection plays the same role for vector bundles as the Kähler-Einstein metric for complex manifolds. Historically, Calabi [6] based his conjectures on the same variational formalism used in Yang-Mills theory. He considered the L^2-norm of the curvature of the metric and showed that under the appropriate assumptions the critial metrics must be Kähler-Einstein. The same reasoning leads from Yang-Mills equations to Hermitian-Yang-Mills connections. Likewise, the existence of a Hermitian-Yang-Mills connection implies the Bogomolov inequality and strengthens the results by giving insight into the case of equality.

The basic problem is then to demonstrate the existence of such connections in stable bundles. This was shown by Donaldson in [9] in the case when M is an algebraic surface. He uses properties of the restriction of the bundle to curves and the theory of secondary characteristic classes. It is difficult to generalize his method directly to higher-dimensional manifolds. These ideas are intertwined with the work of Atiyah-Bott [1] and the convexity of the moment map; see [13]. It is an ultimate goal to extend this entire sphere of ideas to higher dimensions than curves.

In this paper we demonstrate the existence of a Hermitian-Yang-Mills connection in stable holomorphic bundles over any compact Kähler manifold. The method is to perturb the equation and to study the limit of the perturbed equation as the perturbation parameter goes to zero. This reduces the problem to the regularity of a blow-up limit when there is no solution. We study this regularity *via* two avenues: estimates for unitary (real) Yang-Mills type equations on the manifold as a real manifold. Alternatively, we reduce the problem to a regularity question for a weakly holomorphic map. Our solution generalized Hartog's theorem in several variables in that we show that a measurable differentiable function of two complex variables which is separately meromorphic if one fixes almost every point in one variable is in fact meromorphic. As an application, one sees that any weakly holomorphic mapping from the polydisc into an algebraic manifold is in fact a rational mapping. It is very likely that it is also true for Kähler manifolds.

Recently, B. Shiffman has also proved the assertion on meromorphic functions by a very different argument. He applied the assertion to study closed positive (1, 1) currents.

1. Preliminary Discussion

Let E be a holomorphic vector bundle of rank r over a compact Kähler manifold X. For a Hermitian metric h along the fiber of E, the Hermitian connection D: $\Gamma(E) \to \Gamma(T^*X \otimes E)$ is characterized by the properties

(i) $d(h(s,t)) = h(Ds,t) + h(s,Dt)$,

(ii) $D''s = \bar{\partial}s$, where D'' denotes the $(0,1)$ component of the connection D.

We shall often express D as $d + A$, where $A \in \Gamma(\operatorname{End} E \otimes T^*X)$ is the connection matrix. With respect to a local frame $\{e_\alpha\}$, $A = (A_\alpha^\beta)_{1 \leqq \alpha, \beta \leqq n}$ is given by

$$(1.1) \qquad A_\alpha^\beta = (\partial h_{\alpha\bar{\gamma}}) h^{\bar{\gamma}\beta},$$

where $h_{\alpha\bar{\beta}} = h(e_\alpha e_\beta)$ and $(h^{\alpha\bar{\beta}}) = (h_{\alpha\bar{\beta}})^{-1}$.

The curvature $F = dA - A \wedge A$ of the Hermitian connection is a section of $\operatorname{End} E \otimes \Lambda^2 T^*X$. For a holomorphic vector bundle it reduces to

$$(1.2) \qquad F = \bar{\partial} A = h^{-1} \bar{\partial}\partial h + h^{-1}\partial h \wedge h^{-1} \bar{\partial} h.$$

In particular, the Hermitian curvature forms of a holomorphic vector bundle are of type $(1,1)$.

Conversely, the integrability theorem of Newlander-Nirenberg implies that a complex vector bundle admits a holomorphic structure if there exists a $U(r)$ connection whose curvature is of type $(1,1)$.

Given a Kähler metric g on X, we define an operation tr_g: $\Gamma(\operatorname{End} E \otimes T_{1,1}^* X) \to \Gamma(\operatorname{End} E)$ as follows. For a section $F = (F_\alpha^\beta) \in \Gamma(\operatorname{End} E \otimes T_{1,1}^* M)$,

$$(1.3) \qquad \operatorname{tr}_g F = \left(\operatorname{tr}_g F_\alpha^\beta\right)_{1 \leqq \alpha, \beta \leqq n} = \left(\sum g^{j\bar{k}} F_{\alpha j\bar{k}}^\beta\right)_{1 \leqq \alpha, \beta \leqq n},$$

where $F_\alpha^\beta = F_{\alpha j\bar{k}}^\beta \, dz^j \wedge d\bar{z}^k$. From now on we shall consistently use $1 \leqq \alpha, \beta \leqq r$ to indicate fiber indices and $1 \leqq j, k \leqq n$ to denote base indices. We shall also adopt the following convenient notation:

$$(1.4) \qquad \begin{aligned} F_{j\bar{k}} &= \left(F_{\alpha j\bar{k}}^\beta\right)_{1 \leqq \alpha, \beta \leqq r}, \\ \operatorname{tr}_g F &= \sum_{j,k} g^{j\bar{k}} F_{j\bar{k}} \end{aligned}$$

Definition. A holomorphic vector bundle of rank r over a compact Kähler manifold (X, g) is Hermitian-Yang-Mills if there exists a Hermitian metric h for which the Hermitian curvature F satisfies:

$$(1.5) \qquad \operatorname{tr}_g F = \mu I,$$

where I is the identity endomorphism of E and μ is a constant; in local coordinates:

$$(1.5a) \qquad g^{j\bar{k}} F_{\alpha j\bar{k}}^\beta = \mu \delta_\alpha^\beta \quad \text{for all} \quad 1 \leqq \alpha, \beta \leqq r.$$

The Hermitian-Yang-Mills condition (1.5) can also be expressed by first lowering the indices using h, i.e.,

$$F_h = \left(F_{\alpha\bar{\beta}}\right)_{1 \leqq \alpha, \beta \leqq n},$$

where

$$(*) \qquad F_{\alpha\bar\beta} = F_\alpha^\gamma h_{\gamma\bar\beta}.$$

Now contract by g:

$$\operatorname{tr}_g F_h = \Big(\sum_{j,k} g^{j\bar k} F_{\alpha\bar\beta j\bar k}\Big) = \sum g^{j\bar k} F^\gamma_{\alpha j\bar k} h_{\gamma\bar\beta}.$$

Then (1.5) is equivalent to $\operatorname{tr}_g F_h = \mu h$, which is

$$(1.6) \qquad \sum h_{\nu\bar\beta} g^{j\bar k} F^\nu_{\alpha j\bar k} = \mu h_{\alpha\bar\beta}.$$

If we take $E = TX$ to be the holomorphic tangent bundle and $h = g$ ($h_{\nu\bar\beta} = g_{\nu\bar\beta}$) then (1.6) becomes the familiar Kähler-Einstein condition

$$(1.7) \qquad g_{\nu\bar\beta} g^{j\bar k} F^\nu_{\alpha j\bar k} = \mu g_{\alpha\bar\beta} \quad \text{or} \quad \operatorname{Ric} = \mu g.$$

The left-hand side is simply the Ricci curvature

$$F_{\alpha\bar\beta} = g^{j\bar k} F_{\alpha\bar\beta j\bar k}.$$

The Hermitian-Einstein factor μ can be expressed in terms of invariants of E and X. The first Chern form $C_1(E) = C_1(E, h)$ is given by

$$C_1(E) = \frac{i}{2\pi} \sum_{\alpha,\beta} h^{\alpha\bar\beta} F_{\alpha\bar\beta} = \frac{i}{2\pi} \sum h^{\alpha\bar\beta} F_{\alpha\bar\beta j\bar k} dz^j \wedge d\bar z^k.$$

If ω is the Kähler form given by $(i/2\pi) g_{j\bar k}\, dz^j \wedge d\bar z^k$, then

$$C_1(E) \wedge *\omega = \frac{C_1(E) \wedge \omega^{n-1}}{(n-1)!}$$

$$= \Big(\sum h^{\alpha\bar\beta} g^{j\bar k} F_{\alpha\bar\beta j\bar k}\Big) \frac{\omega^n}{n!}.$$

Therefore by (1.6) we have

$$C_1(E) \wedge *\omega = \mu \sum h^{\alpha\bar\beta} h_{\alpha\bar\beta} \frac{\omega^\eta}{\eta!} = \mu (\operatorname{rank} E) \frac{\omega^n}{n!}.$$

Integrating the above, we get

$$(1.8) \qquad \mu = \mu(E) = \frac{1}{\operatorname{vol} X} \frac{\int_X C_1(E) \wedge *\omega}{\operatorname{rank} E} = \frac{1}{(\operatorname{vol} X)} \frac{\deg_\omega E}{\operatorname{rank} E},$$

where

$$\deg_\omega E = \int_X C_1(E) \wedge * \omega. \tag{1.9}$$

Notice that for any Hermitian metrics h and h', we can find a function p such that

$$\int_X [C_1(E,h) - C_1(E,h')] \wedge * \omega = \int_X \partial\bar{\partial} p \wedge * \omega = \int_X d(\bar{\partial} p \wedge * \omega) = 0$$

because $d\omega = 0$ (ω is a Kähler form). Thus $\deg_\omega E$ (hence also μ) is independent of the Hermitian metric h (but does depend on ω).

From now on, we shall call $\mu I - \operatorname{tr}_g F$ the Hermitian-Yang-Mills tensor and denote it by H.

For a coherent analytic subsheaf $\mathfrak{F}$ of E, define

$$C_1(\mathfrak{F}) = C_1(\det \mathfrak{F}^{**}), \quad \deg_\omega = \int_X C_1(\mathfrak{F}) \wedge * \omega \quad \text{and} \quad \mu(\mathfrak{F}) = \deg_\omega \mathfrak{F}/\operatorname{rank} \mathfrak{F}.$$

DEFINITION. A holomorphic vector bundle E over a compact Kähler manifold (X, ω) is (semi)-stable if, for every coherent subsheaf $\mathfrak{F}$ of lower rank, $\mu(\mathfrak{F}) < \mu(E)$(respectively $\leqq$).

Remarks. (i) $\mathfrak{F}^* = \operatorname{Hom}(\mathfrak{F}, \mathcal{O})$, where $\mathcal{O}$ is the structure sheaf of M.

(ii) F is reflexive if and only if $\mathfrak{F}^{**} = (\mathfrak{F}^*)^* = \mathfrak{F}$. A rank one reflexive sheaf is a holomorphic line bundle.

(iii) A reflexive sheaf is locally free (i.e., a holomorphic vector bundle) outside a subvariety of codimension greater than or equal to 2.

(iv) It is enough (in the definition of stability) to consider only reflexive subsheaves.

One has the following:

PROPOSITION (see [19]). *A stable holomorphic vector bundle over a compact projective manifold is simple, i.e.,* $\dim_{\mathbf{C}} \Gamma(M, \operatorname{End} E) = 1$), *namely holomorphic endomorphisms of E are of the form* λId_E, *with* $\lambda \in \mathbf{C}$.

The proof can be sketched as follows. By tensoring E with H^k for appropriate power of the hyperplane section bundle, $\mu(E \otimes H^k) \in \{-r+1, -r+2, \cdots, -1, 0\}$, where $r = \operatorname{rank} E$. Clearly, $E \otimes H^k$ is stable if and only if E is stable and $E \otimes H^k$ is simple if and only if E is simple. Now if $E \otimes H^k$ is not simple, the structure sheaf $\mathcal{O}$ is a subsheaf and $\mu(\mathcal{O}) = 0 \geq \mu(E \otimes H^k)$. This will imply that $E \otimes H^k$ is not stable.

It is very likely that the proposition also holds for compact Kähler manifolds. In any case, we shall assume simplicity is part of the definition of stability.

With this definition, the main theorem of this paper is the following:

THEOREM. *A stable holomorphic vector bundle over a compact Kähler manifold admits a unique Hermitian-Yang-Mills connection.*

As we remarked in the introduction, this generalizes a result of Donaldson's for complex algebraic surfaces. Donaldson uses a heat equation method, secondary characteristic classes, a theorem on the restriction of stable bundles over surfaces to special curves, convergence of connections with L^2-bounded curvatures as unitary connections, and the removable singularities theorem for Yang-Mills equations over real four-manifolds. Our first approach was to try to either extend or simplify his method. It is not hard to see that solutions of the heat equation do exist for all time in any dimension, and one feels that their behavior as $t \to \infty$ is much the same in all dimensions. In fact, our *a priori* estimates in Section 5 do show convergence (as *unitary* connections) off a set of real codimension four. However, the theorem in algebraic geometry on restrictions to curves is missing. Even more serious from the analytic point of view, we know that there is no removable singularities theorem, since we may actually be obtaining sheaf solutions, or solutions which are of necessity singular on a complex submanifold of X of complex codimension two. The analysis of this limit has not been carried out, although we conjecture that there is a "removable singularities" theorem that constructs a sheaf from our limit which is singular off a set of real codimension four.

Our approach is then of necessity different from that of Donaldson. We fix a base metric and its holomorphic structure on E and replace the heat equation by the one-parameter family of nonlinear elliptic equations

$$H = H_0 - g^{\bar{\beta}\alpha}\,\partial_{\bar{\beta}}\left(h^{-1}\,\partial^0_\alpha h\right) = -\varepsilon \log h. \tag{1.10}$$

Here we think of $\varepsilon = e^{-t}$ as replacing the time parameter in the heat equation. Recall that H is the Hermitian-Yang-Mills tensor associated with h. The term $\log h$ is only mysterious at first. Recall we have, point-wise over each point of X, two bilinear forms, the original metric and the new metric h. Therefore, for any smooth function $\rho\colon \mathbb{R}^+ \to \mathbb{R}$ we can talk about $\rho(h)$. As long as h is smooth, so will be $\rho(h)$.

We show that solutions always exist for $0 < \varepsilon < \infty$, and persist to $\varepsilon = 0$ unless a certain very specific disaster happens to the bundle. Analytically, we detect this precisely in the form that, for $\alpha(\varepsilon_j) \to 0$, $\varepsilon_j \to 0$,

$$\alpha(\varepsilon_j)\log h(\varepsilon_j) \underset{j\to 0}{\to} \sum_{j=1}^{J} a_j \pi_j.$$

Here the a_j are positive constants and π_j is the possibly singular orthogonal projection on the regular points of a subsheaf $E_j \subset E$ of rank less than E. Moreover, $E_1 \subset E_2 \subset \cdots \subset E_j$ provides a filtration, rank $E_{j-1} <$ rank E_j. Because of the relationship between the Hermitian-Yang-Mills tensor of E and this filtration, as well as the near orthogonal decomposition (which makes the second fundamental form small) we find

$$\frac{C_1(E_j) \cup \omega^{n-1}}{\text{rank } E_j} \geqq \frac{C_1(E_k) \cup \omega^{n-1}}{\text{rank } E_k}$$

for all $j \geqq k \geqq m$. This of course violates the stability condition. This behavior intuitively corresponds to what happens to unstable bundles under one-parameter subgroups as described by Atiyah and Bott [1].

The technical part of the proof is quite straightforward except for one point. We obtain the E_j as "holomorphic" in a very weak sense from a weak convergence argument. Obtaining enough regularity to describe the E_j as sheaves is more difficult. We have two completely different solutions to this difficulty. One uses estimates for connections and curvatures in the holomorphic bundles considered only as unitary bundles which gives enough regularity to insure smooth convergence off a set of real Hausdorf dimension four. The second independent method is a direct proof of a regularity theorem for π_j, which amounts to showing that a L_1^2-weakly holomorphic map into a projective variety is actually meromorphic.

2. Existence of Solutions to the Perturbed Equations

Let E be a holomorphic vector bundle over a compact Kähler manifold X. Fix the background metric h_0 on E; then for any other metric h_1 on E, we define a smooth endomorphic h on E by

$$\langle s, t\rangle_1 = \langle hs, t\rangle_0. \tag{2.1}$$

In terms of local coordinates,

$$h = h_0^{-1}h_1, \tag{2.2}$$

or

$$h_\alpha^\gamma = \sum_\beta h_0^{\gamma\bar\beta}(h_1)_{\alpha\bar\beta}.$$

Clearly, h is selfadjoint and positive. Hence,

$$\langle hs, t\rangle_0 = \langle s, ht\rangle_0, \tag{2.3}$$

and $\langle hs, s\rangle > 0$ if $s \neq 0$.

Conversely, given any selfadjoint positive endomorphism h, equation (2.1) defines a Hermitian metric on E.

From (2.2) we see that the Hermitian connection forms A^1 and A^0 are related by

$$\begin{aligned} A^1 &= h_1^{-1}\,\partial h_1 \\ &= h^{-1}h_0^{-1}\,\partial(h_0 h) \\ &= h^{-1}A^0 h + h^{-1}\,\partial h \\ &= h^{-1}\big(\partial h + [A^0, h]\big) + A^0 \\ &= A^0 + h^{-1}\,\partial^0 h. \end{aligned}$$

Hence,

$$F^1 = \bar{\partial}A^1 = \bar{\partial}A^0 + \bar{\partial}(h^{-1}\,\partial^0 h).$$

The corresponding Hermitian-Yang-Mills tensors are related by

$$H - H_0 = -\operatorname{tr}_g \bar{\partial}(h^{-1}\,\partial^0 h). \tag{2.4}$$

We simply write (2.4) as

$$H = H_0 - \bar{\partial}(h^{-1}\,\partial^0 h). \tag{2.5}$$

Thus the existence of the Hermitian-Yang-Mills connection is equivalent to the existence of a selfadjoint, positive endomorphism of E satisfying the equation

$$H = H_0 - \bar{\partial}(h^{-1}\,\partial^0 h) = 0. \tag{2.6}$$

Before we proceed, we introduce some notation. Given any real-valued function ϕ, we define $\phi(h)$ as follows: Diagonalize h with respect to h_0 as $h = \Sigma_\alpha e^{\lambda_\alpha} e_\alpha \otimes e_\alpha^*$. Then we define $\phi(h)$ to be $\Sigma_\alpha \phi(e^{\lambda_\alpha}) e_\alpha \otimes e_\alpha^*$. In particular, $\log h = \Sigma_\alpha \lambda_\alpha e_\alpha \otimes e_\alpha^*$.

To prove the main theorem, we study the following perturbed equation:

$$H + \varepsilon \log h = H_0 - \bar{\partial}(h^{-1}\,\partial^0 h) + \varepsilon \log h = 0. \tag{2.7}$$

This equation will be studied in these two sections. The problem of letting $\varepsilon \to 0$ will be studied in Section 4.

Before we proceed, we normalize h_0 conformally so that $\operatorname{tr} H_0 = 0$. This can be done because, by (1.8),

$$\int_X \operatorname{tr} H_0 \frac{\omega^n}{n!} = \deg_\omega E - \mu(\operatorname{rank} E)\operatorname{vol} M = 0.$$

(Note that if $h = ph_0$, then $\operatorname{tr} H(h) = \operatorname{tr} H_0 - (\operatorname{rank} E)\Delta p$. The condition for the existence of p such that $\operatorname{tr} H(h) = 0$ is simply given by $\int_X \operatorname{tr} H_0 = 0$.)

To solve (2.7) for large ε, we set up a continuity argument by solving the one-parameter family of equations:

$$L_{\varepsilon,\sigma}(h) = H_0 - \bar\partial(h^{-1}\partial^0 h) + \varepsilon \log h - \sigma h^{-1/2} H_0 h^{1/2} = 0, \tag{2.8}$$

with

$$0 \leqq \sigma \leqq 1.$$

Let

$$T_\varepsilon = \{\sigma \in [0,1] | L_{\varepsilon,\sigma}(h) = 0 \text{ has a solution}\}. \tag{2.9}$$

Then clearly T_ε is non-empty since, for $\sigma = 1$, $h =$ identity solves the equation for $L_{\varepsilon,1}(h) = 0$. As usual, we shall show that T_ε is both open and closed. This implies $T_\varepsilon = [0,1]$ and $0 \in T_\varepsilon$ which means (2.7) has a solution.

Before we go on, we make the following observation:

Proposition 2.1. *Suppose h solves* (2.8). *Then, under the normalization* $\operatorname{tr} H_0 = 0$, $\det h = 1$.

Proof: Taking the trace of (2.8), we obtain

$$\operatorname{tr} H_0 - \operatorname{tr} \bar\partial(h^{-1}\partial^0 h) + \operatorname{tr}(\varepsilon \log h) - \sigma \operatorname{tr}(h^{-1/2} H_0 h^{1/2}) = 0. \tag{2.10}$$

Since

$$\begin{aligned} \operatorname{tr} \bar\partial(h^{-1}\partial^0 h) &= \operatorname{tr}(\bar\partial(h^{-1}\partial h) + \partial(h^{-1}(A^0 h - hA^0))) \\ &= \partial\bar\partial(\operatorname{tr}\log h), \end{aligned}$$

we have

$$\Delta \operatorname{tr}\log h - \varepsilon \operatorname{tr}\log h = 0.$$

By the maximum principle on compact manifolds, $\operatorname{tr}\log h = 0$, and therefore, $\det h = 1$.

In order to see that T_ε is open, one applies the implicit function theorem and hence one wants to study the linearized operator $\delta L_{\varepsilon,\sigma}$. For any C^1 function ϕ, we compute the variation of $\phi(h)$ by

$$\begin{aligned} \delta\phi(h) &= \sum_\alpha \phi'(e^{\lambda_\alpha}) e^{\lambda_\alpha} \delta\lambda_\alpha e_\alpha \otimes e_\alpha^* \\ &\quad + \sum_\alpha \phi(e^{\lambda_\alpha}) \delta e_\alpha \otimes e_\alpha^* + \sum_\alpha \phi(e^{\lambda_\alpha}) e_\alpha \otimes \delta e_\alpha^*. \end{aligned} \tag{2.11}$$

Let $\delta e_\alpha = \Sigma_\beta \delta a_\alpha^\beta e_\beta$. Then, since $\delta[e_\alpha^*(e_\beta)] = 0$, $\delta e_\alpha^* = -\Sigma \delta a_\beta^\alpha e_\beta^*$. Therefore,

$$\delta\phi(h) = \sum_\alpha \phi'(e^{\lambda_\alpha})\delta\lambda_\alpha e_\alpha \otimes e_\alpha^* + \sum_{\alpha,\beta}\left[\phi(e^{\lambda_\alpha}) - \phi(e^{\lambda_\beta})\right]\delta a_\alpha^\beta e_\beta \otimes e_\alpha^*. \tag{2.12}$$

From now on we shall define the trace inner product in End E by

$$\langle A, B\rangle = \operatorname{tr} AB^*,$$

where the identification $E = E^*$ is made *via* the base metric h_0. All the constants depend only on the Kähler metric on X and the base metric on E.

LEMMA 2.1. *$\langle \delta \log h, \delta h h^{-1}\rangle \geqq (m/(e^m - 1))|\delta h \cdot h^{-1}|^2$, where $m = \max_X |\log h|$.*

Proof: From (2.10) we have

$$\delta \log h = \sum \delta\lambda_\alpha e_\alpha \otimes e_\alpha^* + \sum(\lambda_\alpha - \lambda_\beta)\delta a_\alpha^\beta e_\beta \otimes e_\alpha^*.$$

Furthermore,

$$\delta h \cdot h^{-1} = \sum_\alpha (\delta\lambda_\alpha) e_\alpha \otimes e_\alpha^* + \sum(1 - e^{\lambda_\beta - \lambda_\alpha})\delta a_\alpha^\beta e_\beta \otimes e_\alpha^*.$$

Thus,

$$\langle \delta h h, \delta h \cdot h^{-1}\rangle = \sum_\alpha |\delta\lambda_\alpha|^2 + \sum_{\alpha,\beta}(1 - e^{\lambda_\beta - \lambda_\alpha})(\lambda_\alpha - \lambda_\beta)|\delta a_\alpha^\beta|^2$$

and

$$|\delta h \cdot h^{-1}|^2 = \sum |\delta\lambda_\alpha|^2 + \sum |\delta a_\alpha^\beta|^2 (1 - e^{\lambda_\beta - \lambda_\alpha})^2.$$

It is a simple exercise to check that, for any λ_α, λ_β,

$$\frac{\lambda_\alpha - \lambda_\beta}{1 - e^{\lambda_\beta - \lambda_\alpha}} \geqq \frac{m}{e^m - 1},$$

with $m \geq \max(|\lambda_\alpha|, |\lambda_\beta|)$.

The lemma follows from this fact and the above two inequalities.

LEMMA 2.2.

$$\int_X \langle \delta(h^{-1/2} H_0 h^{1/2}), \delta h \cdot h^{-1}\rangle \leqq 2\|h^{-1/2} H_0 h^{1/2}\|_\infty \int_X |\delta h \cdot h^{-1}|^2.$$

Proof: By computation,

$$\delta\left(h^{-1/2}H_0h^{1/2}\right) = -\left(h^{-1/2}\delta h^{1/2}\right)\left[h^{-1/2}H_0h^{1/2}\right] + \left[h^{-1/2}H_0h^{1/2}\right]h^{-1/2}\delta h^{1/2}.$$

On the other hand,

$$h^{-1/2}\delta h^{1/2} = \tfrac{1}{2}\sum\delta\lambda_\alpha e_\alpha\otimes e_\alpha^* + \sum\delta a_\alpha^\beta\left(1-\exp\left\{\tfrac{1}{2}(\lambda_\beta-\lambda_\alpha)\right\}\right)e_\beta\otimes e_\alpha^*.$$

Hence,

$$|h^{-1/2}\delta h^{1/2}|^2 = \tfrac{1}{4}\sum_\alpha|\delta\lambda_\alpha|^2 + \sum|\delta a_\alpha^\beta|^2\left(1-\exp\left\{\tfrac{1}{2}(\lambda_\beta-\lambda_\alpha)\right\}\right)^2 \leqq |\delta h\cdot h^{-1}|^2,$$

and Lemma 2.2 follows.

LEMMA 2.3. $\int_X\langle\delta(\bar\partial(h^{-1}\,\partial^0h)),(\delta h)\cdot h^{-1}\rangle \geqq e^{-2m}\int_X|\partial^0(\delta h\cdot h^{-1})|^2$.

Proof: By computation,

$$\begin{aligned}\delta\left(\bar\partial\left(h^{-1}\,\partial^0h\right)\right) &= \bar\partial\left(-h^{-1}\delta h\cdot h^{-1}\,\partial^0h + h^{-1}\,\partial^0(\delta h)\right)\\ &= \bar\partial\left(-h^{-1}\delta h\,\partial^0(h^{-1})\cdot h + h^{-1}\,\partial^0(\delta h)\right)\\ &= \bar\partial\left(h^{-1}\,\partial^0(\delta h\cdot h^{-1})h\right).\end{aligned}$$

Therefore,

$$\begin{aligned}-\int_X\langle\delta\left(\bar\partial(h^{-1}\,\partial^0h)\right),\delta h\cdot h^{-1}\rangle &= -\int_X\langle\bar\partial\left(h^{-1}\,\partial^0(\delta h\cdot h^{-1})h\right),\delta h\cdot h^{-1}\rangle\\ &= \int_X\langle h^{-1}\,\partial^0(\delta h\cdot h^{-1})h,\,\partial^0(\delta h\cdot h^{-1})\rangle\\ &\geqq e^{-2m}\int_X|\partial^0(\delta h\cdot h^{-1})|^2.\end{aligned}$$

PROPOSITION 2.2. *Let* $\phi = \delta h$ *and* $m = \max|\log h|$. *Then,*

$$\int_X\langle\delta L_{\varepsilon,\delta}(h)(\phi),\phi h^{-1}\rangle$$

$$\geqq e^{-2m}\int_X|\partial^0(\phi\cdot h^{-1})|^2 + \frac{\varepsilon m}{e^m-1}\int_X|\phi\cdot h^{-1}|^2 - 2\sigma e^{2m}\max|H_0|\int_X|\phi\cdot h^{-1}|^2$$

Proof: By computation,

$$\delta L_{\varepsilon,\sigma}(h)(\phi) = -\operatorname{tr}\bar\partial\delta(h^{-1}\,\partial^0 h) + \varepsilon\delta(\log h) - \sigma\delta\big(h^{-1/2}H_0h^{1/2}\big)$$
$$= \operatorname{tr}\bar\partial\big(h^{-1}\,\partial^0(\phi\cdot h^{-1})h\big) + \varepsilon\delta(\log h) - \sigma\delta\big(h^{-1/2}H_0h^{1/2}\big).$$

The proposition follows from Lemma 2.1, Lemma 2.2 and Lemma 2.3.

Using the normalization $\operatorname{tr} H_0 = 0$, we prove in Proposition 2.1 that $\det h = 1$ and $\operatorname{tr}\delta h\cdot h^{-1} = 0$. Therefore, $\langle \delta h\cdot h^{-1}, I\rangle = 0$.

Since the kernel of the operator $\bar\partial\partial^0$ gives holomorphic sections of End E and E is stable, it consists of a constant multiple of the identity. Therefore, if $K > 0$ is the first non-zero eigenvalue of the operator $-\bar\partial^0\partial^0$,

$$\int_X |\partial^0(\phi h^{-1})|^2 \geqq K\int_X |\phi h^{-1}|^2.$$

Hence we have proved the following:

Proposition 2.3. *Let* $\max_X|\log h| \leqq m$ *and* $\phi = \delta h$ *with* $\operatorname{tr}\phi h^{-1} = 0$. *Then,*

$$\int_X \langle \delta L_{\varepsilon,\sigma}(h)\phi, \phi h^{-1}\rangle \geqq C(m,\varepsilon,\sigma)\int_X |\phi\cdot h^{-1}|^2,$$

where

$$C(m,\varepsilon,\sigma) = e^{-2m}K + m(e^m - 1)^{-1}\varepsilon - \sigma e^{2m}\max_X|H_0|.$$

In order to make use of Proposition 2.3, we need an estimate on m.

Lemma 2.4. *Suppose* $H_0 - \bar\partial(h^{-1}\,\partial^0 h) + \varepsilon\log h - h^{-1/2}\tilde H_0 h^{1/2} = 0$ *and* $u = \log h$. *Then,*

$$|u||H_0 - \tilde H_0| \geqq -\tfrac12\Delta|u|^2 + \varepsilon|u|^2,$$

and

$$m = \max_X|u| \leqq \varepsilon^{-1}\max_X|H_0 - \tilde H_0|.$$

Proof: Since $\langle u, h^{-1/2}\tilde H_0 h^{1/2}\rangle = \langle u, e^{-u/2}\tilde H_0 e^{u/2}\rangle = \langle u, \tilde H_0\rangle$, it suffices to prove that

$$\langle u, -\bar\partial(h^{-1}\,\partial h) + \varepsilon\log h\rangle \geqq -\tfrac12\Delta|u|^2 + \varepsilon|u|^2.$$

It is also sufficient to prove this inequality for a dense set of $u \in C^\infty(\text{End } E)$. Therefore we can assume u has distinct eigenvalues on an open dense set of full measure. On this open set, we can write

$$u = \sum_\alpha \lambda_\alpha e_\alpha \otimes e_\alpha^*.$$

Further,

$$h = \sum_\alpha e^{\lambda_\alpha} e_\alpha \otimes e_\alpha^*.$$

Let $A_i^{\alpha\beta} = \langle \partial_i^0 e_\alpha, e_\beta \rangle$. Then,

$$\partial_i^0 h = \sum e^{\lambda_\alpha}(\partial_i \lambda_\alpha) e_\alpha \otimes e_\alpha^* + \sum (e^{\lambda_\alpha} - e^{\lambda_\beta}) A_i^{\alpha\beta} e_\beta \otimes e_\alpha^*,$$

and

$$\partial_i^0 u = \sum (\partial_i \lambda_\alpha) e_\alpha \otimes e_\alpha^* + \sum (\lambda_\alpha - \lambda_\beta) A_i^{\alpha\beta} e_\beta \otimes e_\alpha^*.$$

Therefore,

$$\begin{aligned}\langle \partial^0 u, h^{-1} \partial^0 h \rangle &= \sum |\partial \lambda_\alpha|^2 + \sum |A^{\alpha\beta}|^2 (e^{\lambda_\alpha - \lambda_\beta} - 1)(\lambda_\alpha - \lambda_\beta) + \varepsilon |u|^2 \\ &\geqq \sum |\partial \lambda_\alpha|^2.\end{aligned}$$

Further,

$$\begin{aligned}\langle u, -\bar\partial(h^{-1} \partial^0 h) + \varepsilon \log h \rangle &= -\partial \langle u, h^{-1} \partial^0 h \rangle + \langle \partial^0 u, h^{-1} \partial^0 h \rangle + \varepsilon |u|^2 \\ &\geqq -\tfrac{1}{2} \Delta |u|^2 + \varepsilon |u|^2.\end{aligned}$$

The estimate on $\max_X |u|$ follows from the maximum principle.

By inverting the operator $(-\frac{1}{2}\Delta + \varepsilon)$, we can also derive the following:

COROLLARY 2.1. *If $\int_X |u|^2 \leqq m^2$, then*

$$\max_X |u| \leqq C(m + \max |H_0 - \tilde{H}_0|),$$

where C is independent of ε.

Using Lemma 2.4, we see that the constant $C(M, \varepsilon, \sigma)$ in Proposition 2.3 can be estimated in terms of K, $|H_0|$, σ and ε. It is clear that when ε is large (so that m is small) compared with $\max |H_0|$, $C(m, \varepsilon, \sigma) > 0$. The same conclusion holds when $\sigma = 0$ and $\varepsilon > 0$. This shows that when ε is large or when $\sigma = 0$, the operator $\delta L_{\varepsilon, \sigma}(h)$ is invertible.

Let h_0 be a solution of the equation $L_{\varepsilon, \sigma}(h) = 0$. Then the map $h \to L_{\varepsilon, \sigma}(h)$ can be considered as a nonlinear smooth map from L_k^p-space of selfadjoint operators of E to L^p_{k-2}-space of selfadjoint operators of E. (Note that H is Hermitian with respect to the metric defined by h.) It is easy to check that the openness of T_ε follows if one can prove that the above map maps a small neighborhood of h_0 to a small neighborhood of the zero operator. By the implicit function theorem, it is enough to prove the surjectivity of the linearized operator $\delta L_{\varepsilon,\sigma}(h_0)$. The surjectivity follows by showing that the kernel of its adjoint operator is trivial.

Our second step is to show that T_ε is closed when ε is large. This will require *a priori* estimates. Under the assumption that ε is large, Proposition 2.3 implies that $m = \sup_X |\log h|$ is small. Therefore the closedness of T_ε follows from

PROPOSITION 2.4. *If* $H_0 - \bar\partial(h^{-1}\partial^0 h) + \varepsilon \log h - h^{-1/2}\tilde H_0 h^{1/2} = 0$ *and if* $m = \max_X |\log h|$ *is small, then*

$$\|h\|_{C^k(X)} \leqq C\big(m, \|H_0\|_{C^k(X)}, \|\tilde H_0\|_{C^k(X)}\big).$$

Proof: Let $u = \log h$. Then we can rewrite the above equation as

$$-\bar\partial\big(e^{-u}F(u)\partial^0 u\big) + \varepsilon u = -H_0 + e^{-u/2}\tilde H_0 e^{u/2},$$

where $F(u) = \delta e^u/\delta u$ is the matrix of derivatives of exp at u. Since $m = \max_X |\log h|$ is small, $|I - e^{-u}F(u)| = \delta$ is small.

Let $\Delta_0 = \bar\partial\partial_0$ and $I - e^{-u}F(u) = G(u)$. Then,

$$-\Delta_0 u + u = (1-\varepsilon)u + \bar\partial\big(G(u)\partial^0 u\big) - H_0 + e^{-u/2}\tilde H_0 e^{u/2}.$$

Since $-\Delta_0 + I$ is an invertible operator from L^p_1 to L^p_{-1}, there is a constant $C(p)$ such that

$$\|u\|_{L^p_1} \leqq C(p)\big[(1-\varepsilon)\|u\|_{L^p_{-1}} + \delta\|u\|_{L^p_1} + \max|H_0| + e^{2m}\max|\tilde H_0|\big].$$

Here we estimated the term

$$\|\bar\partial\big(G(u)\partial^0 u\big)\|_{L^p_{-1}} \leqq \|G(u)\partial^0 u\|_{L^p} \leqq \delta\|u\|_{L^p_1}.$$

Therefore, if $1 - C(p)\delta \geqq \frac{1}{2}$ or if δ is sufficiently small, we have a uniform L^p_1-estimate on u. Exactly the same argument provides L^p_k-estimates for u and hence the proposition.

Proposition 2.4 shows that T_ε is closed when ε is large. In particular, $\sigma = 0$ is a point in T_ε. In the next section we shall prove the existence of solution for $L_{\varepsilon,0}(h) = 0$ for all $\varepsilon > 0$.

3. Existence of Solution for $L_\varepsilon(h) = 0$ for $\varepsilon > 0$

In this section we shall prove existence of a smooth solution for $L_\varepsilon(h) = 0$ where $L_\varepsilon = L_{\varepsilon,0}$. From the previous section we already know the existence for ε large. We also prove that the operator δL_ε is invertible. Hence, the set of

$0 < \varepsilon < \infty$ where $L_\varepsilon(h) = 0$ has a solution is a non-empty open set. We shall prove that it is also closed and hence $L_\varepsilon(h) = 0$ always has a solution for $\varepsilon > 0$.

Since the set of ε for which $L_\varepsilon(h) = 0$ is open, we can differentiate the equation with respect to ε and obtain $\delta L_\varepsilon(h)(\delta h) + \log h = 0$. We shall find an estimate for δh.

PROPOSITION 3.1. *Suppose* $\delta L_\varepsilon(h)(\delta h) + \log h = 0$. *Then, for* $\Psi = h^{-1/2}(\delta h)h^{-1/2}$,

$$-\Delta|\Psi|^2 + \varepsilon|\Psi|^2 \leqq -2\langle \log h, \Psi\rangle. \tag{3.1}$$

Proof: Let

$$\partial^{\bar{A}} = Ad h^{1/2} \cdot \bar{\partial} \cdot Ad h^{-1/2}$$

and

$$\partial^{A} = Ad h^{-1/2} \cdot \partial^{0} \cdot Ad h^{1/2}.$$

Then since $\delta H = -\bar{\partial}(h^{-1}\partial^0(\delta h h^{-1})h)$, one verifies that

$$\delta H = -h^{-1/2}\big(\partial^{\bar{A}}\partial^{A}(h^{-1/2}\delta h h^{-1/2})\big)h^{1/2}. \tag{3.2}$$

Hence,

$$\partial^{\bar{A}}\partial^{A}\Psi + \partial^{A}\partial^{\bar{A}}\Psi = -\big(h^{-1/2}\delta H h^{1/2} + h^{1/2}\delta H h^{-1/2}\big). \tag{3.3}$$

Since $\delta L_\varepsilon(h)(\delta h) + \log h = 0$, we obtain

$$\begin{aligned}&\partial^{\bar{A}}\partial^{A}\Psi + \partial^{A}\partial^{\bar{A}}\Psi + \varepsilon\big(h^{-1/2}\delta \log h\, h^{1/2} + h^{1/2}\delta \log h\, h^{-1/2}\big)\\ &\quad = -2\log h.\end{aligned} \tag{3.4}$$

Taking the inner product with Ψ, and using the inequalities

$$2\langle \partial^{A}\partial^{\bar{A}}\Psi + \partial^{\bar{A}}\partial^{A}\Psi, \Psi\rangle \leqq \Delta|\Psi|^2, \tag{3.5}$$

and

$$\begin{aligned}&\langle h^{-1/2}(\delta \log h)h^{1/2} + h^{1/2}(\delta \log h)h^{-1/2}, \Psi\rangle\\ &\quad = \langle \delta \log h, \delta h \cdot h^{-1} + h^{-1}\delta h\rangle\\ &\quad \geqq |\Psi|^2,\end{aligned} \tag{3.6}$$

we can derive (3.1) from (3.4).

COROLLARY 3.1. *If* $\delta L_\varepsilon(h)(\delta h) + \log h = 0$ *and* $m = \max_X|\log h|$, *then* $\max_X|\delta h| \leqq C(m)$.

Proof: By the proposition, we can estimate $\max_X(\Psi)$ in terms of $m + (\int_X|\Psi|^2)^{1/2}$. Therefore, it suffices to estimate

$$\int_X|\Psi|^2 = \int_X|\delta h \cdot h^{-1}|^2.$$

It follows from Proposition 2.3 that we can estimate $\int_X|\delta h \cdot h^{-1}|^2$ in terms of m.

PROPOSITION 3.2. *If* $\delta L_\varepsilon(h)(\delta h) + \log h = 0$, *then*

$$\|\delta h\|_{L_2^p} \leqq C(m)\big(\|h\|_{L_2^p} + 1\big). \tag{3.7}$$

Proof: By direct calculation,

$$\begin{aligned}\delta H = \delta h \cdot h^{-1}(H - H_0) - h\big(\bar\partial\partial^0(\delta h) - \bar\partial(\delta H)h^{-1}\,\partial^0 h \\ - \bar\partial h h^{-1}\,\partial^0(\delta h) + \bar\partial h h^{-1}(\delta h)h^{-1}\,\partial^0 h\big).\end{aligned} \tag{3.8}$$

On the other hand,

$$\delta H = -\varepsilon\delta(\log h) - \log h. \tag{3.9}$$

It follows that we have an equation for $\bar\partial\partial^0(\delta h)$. By the preceding corollary, $|\delta h|$ and $|\delta \log h|$ can be estimated by a function of m. Therefore, we have

$$\|\bar\partial\partial^0(\delta h)\|_{L^p} \leqq 2e^{-m}\|\delta h\|_{L_1^{2p}}\|h\|_{L_1^{2p}} + C_1(m)\|h\|_{L_1^{2p}} + C_2(m). \tag{3.10}$$

Since the operator $-\bar\partial\partial^0 + I$ is invertible, we obtain

$$\|\delta h\|_{L_2^p} \leqq C_3(m)\big(\|\delta h\|_{L_1^{2p}}\|h\|_{L_1^{2p}} + \|h\|^2_{L_1^{2p}} + 1\big). \tag{3.11}$$

On the other hand, by interpolation,

$$\|\delta h\|^2_{L_1^{2p}} \leqq \|\delta h\|_{L_2^p}\|\delta h\|_{L^\infty},$$

for p large enough.

Since we have a bound on $|\delta h|$ and $|h|$ in terms of m, we can substitute (3.11) into (3.10) and apply Hölder's inequality to provide a proof of (3.7).

COROLLARY 3.2. *If there exists a one-parameter solution to* $L_\varepsilon(h_\varepsilon) = 0$ *in an interval* $(\varepsilon_0, \varepsilon_0')$ *such that* $|\log h_\varepsilon| \leq m$ *uniformly in* ε, *then*

$$\|h_\varepsilon\|_{L_2^p} \leqq C_4(m)\big(\|h_{\varepsilon_0'}\|_{L_2^p} + 1\big),$$

for all ε *in* $(\varepsilon_0, \varepsilon_0']$.

Proof: Let $f(\varepsilon) = \|h_\varepsilon\|_{L_2^p}$. Then the proposition gives an inequality:

$$|f'(\varepsilon)| \leqq C(m)(f(\varepsilon) + 1). \tag{3.12}$$

The corollary follows by integrating this inequality.

By Lemma 2.4 and Corollary 2.1, we have then proved the following:

THEOREM 3.1. *There exists a solution to the equation $L_\varepsilon(h) = 0$ for all $\varepsilon > 0$. Moreover, if $\int_X |\log h_\varepsilon|^2$ stays bounded for $\varepsilon \to 0$, then there exists a solution in an interval in $(-\varepsilon_0, \infty)$ for some $\varepsilon_0 > 0$.*

For later purpose, we shall need

PROPOSITION 3.3. *If h solves $L_\varepsilon h = 0$, and F_h is the full curvature tensor for the connection associated with h, then*

$$\int_X |F_h|_h^2 \leqq \int_X |F_0|^2,$$

where F_0 is the full curvature tensor of the base metric.

Proof: We have

$$\int_X |F_0|^2 - \int_X |F_h|_h^2 = \int_X |H_0|^2 - \int_X |H|_h^2. \tag{3.13}$$

Since $H + \varepsilon \log h = 0$,

$$|H|_h^2 = \operatorname{tr}(hHh^{-1}H^*) = \operatorname{tr}(HH^*) = |H|^2. \tag{3.14}$$

Therefore we need to show

$$\int_X |H_0|^2 \geqq \int_X |H|^2. \tag{3.15}$$

But

$$\begin{aligned} \int_X \langle H_0 - H, H\rangle &= -\varepsilon \int_X \langle \log h, \bar{\partial}(h^{-1}\partial^0 h)\rangle \\ &= \varepsilon \int_X \langle \partial^0 \log h, h^{-1}\partial^0 h\rangle \geqq 0, \end{aligned} \tag{3.16}$$

and (3.15) follows by the Schwartz inequality.

4. Proof of the Theorem

In order to prove the theorem, we have to prove that if h_ε does not converge, we can produce a subsheaf of E which violates the condition of stability. The subsheaf will be a subbundle outside a closed subvariety of codimension greater than or equal to 2.

The base Hermitian metric allows us to identify a subbundle of E with the Hermitian projection onto that subbundle. Hence it will be a section of End E given by π so that $\pi^2 = \pi = \pi^*$. The holomorphic condition can be identified as

$$(I - \pi)\bar{\partial}\pi = 0. \tag{4.1}$$

This is easily seen to be equivalent to the fact that, for any section f of the subbundle, $\partial_{\bar{\alpha}} f$ is still in the subbundle.

Hence we define a weakly holomorphic subbundle to be a section $\pi \in L_1^2(E, E)$ such that

$$\pi^2 = \pi = \pi^*, \qquad (I - \pi)\bar{\partial}\pi = 0. \tag{4.2}$$

We shall prove that π is actually smooth outside a subvariety of complex codimension greater than or equal to 2 and that it defines a coherent holomorphic subsheaf of E. We shall also prove that if h_ε does not converge, we can normalize in a suitable way and obtain a subsequence converging to such a weakly holomorphic subbundle of E.

We shall use the following lemmas:

Lemma 4.1. *For $0 < \sigma \leqq 1$ and $L_\varepsilon(h) = 0$, we have*

$$|h^{-\sigma/2}\partial^0 h^\sigma|^2 + \varepsilon\langle u, h^\sigma\rangle - \frac{1}{\sigma}\Delta|h^\sigma| \leqq -\langle H^0, h^\sigma\rangle, \tag{4.3}$$

where $h^\sigma = e^{\sigma u}$.

Proof: From $\langle L_\varepsilon h, h^\sigma\rangle = 0$ we obtain

$$\langle h^{-1}\,\partial^0 h, \partial^0 h^\sigma\rangle + \varepsilon\langle u, h^\sigma\rangle - \bar{\partial}\langle h^{-1}\,\partial^0 h, h^\sigma\rangle = -\langle H_0, h^\sigma\rangle. \tag{4.4}$$

By computation,

$$\bar{\partial}\langle h^{-1}\,\partial^0 h, h^\sigma\rangle = \sum_j \bar{\partial}\left(e^{\sigma\lambda_j}\partial\lambda_j\right) = \frac{1}{\sigma}\Delta\cdot\left(\sum_j e^{\sigma\lambda_j}\right), \tag{4.5}$$

$$\begin{aligned}\langle h^{-1}\,\partial^0 h, \partial^0 h^\sigma\rangle &= \sigma\sum_j e^{\sigma\lambda_j}|\partial\lambda_j|^2 + \sum_{k\neq j}|A^{jk}|^2\left(e^{\lambda_k-\lambda_j} - 1\right)\left(e^{\sigma\lambda_k} - e^{\sigma\lambda_j}\right)\\ &\geqq \sigma^2\sum_j e^{\sigma\lambda_j}|\partial\lambda_j|^2 + \sum_{k\neq j}|A^{jk}|^2 e^{-\sigma\lambda_k}\left(e^{\sigma\lambda_j} - e^{\sigma\lambda_k}\right)^2 \\ &= |h^{-\sigma/2}\partial^0 h^\sigma|^2.\end{aligned} \tag{4.6}$$

Note that we use the following elementary inequality:

$$
\begin{aligned}
(e^{\mu}-1)(e^{\sigma\mu+\sigma\lambda}-e^{\sigma\lambda}) &= e^{\sigma\lambda}(e^{\mu}-1)(e^{\sigma\mu}-1) \\
&\geqq e^{\sigma\lambda}(e^{\sigma\mu}-1)^2 \qquad (4.7)\\
&= e^{-\sigma\lambda}(e^{\sigma\mu+\sigma\lambda}-e^{\sigma\lambda})^2,
\end{aligned}
$$

where $0<\sigma\leqq 1$, λ is arbitrary. In the application, we set $\lambda=\lambda_j$ and $\mu=\lambda_j-\lambda_k$.

Proposition 4.1. *Suppose that* $\overline{\lim}_{\varepsilon\to 0}\int_X|\log h_\varepsilon|^2=\infty$. *Then there exists a subsequence* $\varepsilon_l\to 0$ *and a normalizing constant* $\rho_l\to 0$ *such that, in* L_1^2,

$$
\begin{aligned}
h_l &= \rho_l h(\varepsilon_l)\leqq I \qquad (\text{as a metric}),\\
h_l &= \rho_l h(\varepsilon_l)\rightharpoonup h_\infty\neq 0.
\end{aligned}
\qquad (4.8)
$$

Finally, $I-\lim_{\sigma\to 0}h_\infty^\sigma$ *represents a weakly holomorphic subbundle.*

Proof: Let $\rho(\varepsilon)=\exp\{-M(\varepsilon)\}$, where $M(\varepsilon)$ is the maximum of the largest eigenvalue of $\log h(\varepsilon)$. Clearly, since $\operatorname{tr}\log h(\varepsilon)=0$, $M(\varepsilon)$ has the same order as $m(\varepsilon)$.

Then $\rho(\varepsilon)h(\varepsilon)\leqq I$. From Lemma 4.1 and Lemma 2.4 we have

$$
-\Delta|h(\varepsilon)|\leqq 2\max_X|H^0|\,|h(\varepsilon)|. \qquad (4.9)
$$

By applying the maximum principle, we can estimate $\max_X\rho(\varepsilon)|h(\varepsilon)|$ in terms of $(\int_X|\rho(\varepsilon)h(\varepsilon)|^2)^{1/2}$. Therefore, $\int_X|\rho(\varepsilon)h(\varepsilon)|^2$ is bounded from below by a constant independent of ε. This guarantees that any weak L_1^2-limit of any subsequence of $\rho(\varepsilon)h(\varepsilon)$ is nontrivial (as it implies strong convergence in L^2).

In Lemma 4.1, we can replace h by ρh and hence obtain, for $\sigma=1$,

$$
\begin{aligned}
\int_X|d(\rho(\varepsilon)h(\varepsilon))|^2 &= 2\int_X|\partial^0(\rho(\varepsilon)h(\varepsilon))|^2\\
&\leqq 2\int_X|(\rho(\varepsilon)h(\varepsilon))^{-1/2}\partial^0(\rho(\varepsilon)h(\varepsilon))|^2 \qquad (4.10)\\
&\leqq 4\max_X|H^0|\operatorname{vol}(X).
\end{aligned}
$$

Therefore, for some subsequence of $\varepsilon_l\to 0$, $\rho_l=\rho(\varepsilon_l)\to 0$ and $h_l=\rho_l h_l(\varepsilon_l)$ has a non-zero weak limit h_∞.

The same argument as in (4.10) also shows that $\int_X |dh_l^\sigma|^2$ is bounded by $2\max_X |H^0| \mathrm{vol}(X)$. By taking a diagonal process, we can assume that h_l^σ converges to h_∞^σ almost everywhere and hence h_l^σ converges weakly to h_∞^σ in L_1^2.

The uniform bound on the L_1^2-norm of h_l^σ gives the same bound on h_∞^σ for all σ. It then follows that $I - h_\infty^\sigma$ has a weak limit in L_1^2 for some subsequence $\sigma \to 0$. We call the limit π.

It is easy to show that $\pi = \pi^* = \pi^2$ almost everywhere. It remains to prove the holomorphic condition. Note that $\pi(I - \pi) = 0$ as a distribution; so

$$|(I - \pi)\bar{\partial}\pi|^2 = |\bar{\partial}(I - \pi)\pi|^2 = |\pi\partial^0(I - \pi)|^2.$$

In order to show that the later quantity is zero, we notice that, for $0 \leqq \lambda \leqq 1$,

$$\lambda^{-\sigma} \geqq \frac{2s + \sigma}{s}(1 - \lambda^s) \geqq 0. \tag{4.11}$$

Hence,

$$h^{-\sigma/2} \geqq \frac{2s + \frac{1}{2}\sigma}{s}(I - h_l^s). \tag{4.12}$$

It follows from Lemma 4.1 and this inequality that

$$\begin{aligned} \int_X |(I - h_l^s)\partial^0 h_l^\sigma|^2 &\leqq \left(\frac{s}{2s + \frac{1}{2}\sigma}\right)^2 \int_X |h_l^{-\sigma/2}\partial^0 h_l^\sigma|^2 \\ &\leqq \left(\frac{s}{2s + \frac{1}{2}\sigma}\right)^2 \max_X |H_0|. \end{aligned} \tag{4.13}$$

Hence for $l \to \infty$, we have

$$\int_X |(I - h_\infty^s)\partial^0 h_\infty^\sigma|^2 \leqq \left(\frac{s}{2s + \sigma}\right)^2 \max|H_0|. \tag{4.14}$$

If we first let $s \to 0$ and then $\sigma \to 0$, we obtain

$$\int_X |\pi\partial^0(I - \pi)|^2 = 0. \tag{4.15}$$

In the next sections we shall show that π represents a holomorphic subsheaf F of E. Here we derive some information about F.

LEMMA 4.2. $0 < \operatorname{rank} F < \operatorname{rank} E$.

Proof: By Section 3, we can assume $\rho_l \to 0$. Since $\operatorname{tr}\log h(\varepsilon_l) = 0$, at least one eigenvalue of $h_l = \rho_l h(\varepsilon_l)$ is smaller than ρ_l. Therefore, h_∞ has at least a one-dimensional kernel and $0 < \operatorname{rank} h_\infty < \operatorname{rank} E$.

The lemma follows from the fact that

$$\operatorname{rank} h_\infty = \operatorname{rank} E - \operatorname{rank} F.$$

PROPOSITION 4.2. $[C_1(F)]/\operatorname{rank} F \geqq [C_1(E)]/\operatorname{rank} E$.

Proof: We compute *via* the Chern-Weil formula. Since the regular points of F are the complements of a complex variety of codimension two, the singular set cannot affect the computation of C_1. In other words, $[C_1(F)] = [C_1(F|\text{regular points})]$. We have the usual formula

$$(4.16)\qquad \begin{aligned} 2\pi[C_1(F)] &= \int_X g^{\alpha\bar\beta}(F)\operatorname{tr} F^0_{\alpha\bar\beta}(F) \\ &= \int_X g^{\alpha\bar\beta}\operatorname{tr}\left(F^0_{\bar\beta\alpha}(E)\pi\right) - \int_X |\pi^\perp\partial^0\pi|^2, \end{aligned}$$

where $F^0_{\alpha\bar\beta}(F)$ is the curvature of the connection included by the base metric on the regular point of $F \subseteq E$, π: $E \to F$ is the Hermitian projection on the regular points of F in this metric and $\pi^\perp\partial^0\pi$: $F \to F^\perp$ is the second fundamental form. Note that $\pi\partial^0\pi = 0$. Therefore, $\pi^\perp\partial^0\pi = \partial^0\pi$. Now,

$$(4.17)\qquad \begin{aligned} \int_X g^{\alpha\bar\beta}\operatorname{tr}\left(F_{\alpha\bar\beta}\pi\right) &= \int_X \operatorname{tr}((H_0 + \mu I)\pi) \\ &= \int_X \operatorname{tr}(H_0\pi) + \frac{\operatorname{rank} F}{\operatorname{rank} E}2\pi[C_1(E)]. \end{aligned}$$

Proposition 4.2 follows from (4.15), (4.16) and the following inequality:

$$(4.18)\qquad \int_X \operatorname{tr}(H_0\pi) \geqq \int_X |\partial^0\pi|^2,$$

which can be proved as follows:

Since $\pi = \lim_{\sigma\to 0}\lim_{l\to\infty}(I - h_l^\sigma)$ weakly in L_1^2 and $\int_X \operatorname{tr} H_0 = 0$, we have

$$(4.19)\qquad \int_X \operatorname{tr}(H_0\pi) = -\lim_{\sigma\to 0}\lim_{l\to\infty}\int_X \operatorname{tr}(H_0 h_l^\sigma).$$

If H is the Hermitian-Yang-Mills tensor associated with the metric $h(\varepsilon_l) = h_l/\rho_l$, then $H_l = -\varepsilon_l \log h(\varepsilon_l)$ and

$$(4.20)\qquad H_l = H_0 - \bar\partial\left(h_l^{-1}\,\partial^0 h_l\right).$$

Therefore,

$$-\int_X \operatorname{tr}(H_0 h_l^\sigma) = -\int_X \operatorname{tr}(H_l h_l^\sigma) - \int_X \operatorname{tr}\left(\bar\partial\left(h_l^{-1}\partial^0 h_l\right)h_l^\sigma\right)$$

$$= \varepsilon_l \int \operatorname{tr}(\log h(\varepsilon_l) h_l^\sigma) + \int \langle h_l^{-1}\,\partial^0 h_l,\, \partial^0 h_l^\sigma\rangle.$$

As $\operatorname{tr}\log h(\varepsilon_l) = 0$ and $h_l^\sigma = \rho_l^\sigma h_l^\sigma(\varepsilon_l))$, $\operatorname{tr}(\log h(\varepsilon_l)h_l^\sigma) > 0$. Therefore, as in Proposition 4.1,

$$-\int_X \operatorname{tr}(H_0 h_l^\sigma) \geqq \int_X \langle \partial^0 h_l^\sigma, \partial^0 h_l^\sigma\rangle. \tag{4.21}$$

The inequality (4.17) now follows from (4.19), (4.10) and the lower semicontinuity

$$\int_X |\partial^0 \pi|^2 \leqq \varliminf_{\substack{\sigma\to 0\\ l\to\infty}} \int |\partial^0 h_l^\sigma|^2.$$

THEOREM 4.1 (Main Theorem). *If E is a stable holomorphic vector bundle over a compact Kähler manifold X, then E admits a Hermitian-Yang-Mills connection.*

Proof: By the results of Section 3, there exists a solution $h = h(\varepsilon)$ to the equation $H = \varepsilon \log h$ for $\varepsilon > 0$. Furthermore, if $\int_X |\log h|^2$ has an upper bound independent of ε, then $h(\varepsilon)$ converges to a smooth Hermitian-Yang-Mills connection as $\varepsilon \to 0$.

If $\int_X |\log h|^2$ does not remain bounded, then Proposition 4.1 gives a weakly holomorphic subbundle π which represents a subsheaf of rank strictly between zero and rank E. Proposition 4.1 shows that it violates the stability condition.

APPENDIX. For later purposes it may be desirable to know more precisely the behavior of h_ε when $\varepsilon \to 0$. This can be described in the following theorem for which we give a semi-rigorous proof.

THEOREM 4.2. *Assume that $h(\varepsilon)$ solves $H + \varepsilon \log h = 0$ and that $\overline{\lim}_{\varepsilon\to\infty} \int_X |\log h(\varepsilon)|^2\, d\mu \to \infty$. Then there exist a subsequence $\varepsilon(l) \to 0$ and a $\delta(l) \to 0$ in $\mathbb{R}^+$ such that in $L_1^2(L(E, E))$ we have the weak convergence:*

$$u_\varepsilon = \delta(l)\log h(\varepsilon(l)) \to u_\infty \neq 0.$$

Moreover, we have u_∞ expressed as a sum:

$$u_\infty = \sum_{j=1}^{J} a_j \pi_j,$$

where $J \geqq 2$, $a_j > a_{j-1}$, and π_j are weak projections satisfying $\pi_j \pi_k = \delta_{jk}\pi_j$ and $\sum_{j=1}^{J}\pi_j = I$. In addition, $\sum_{j \leqq k}\pi_j$ is weakly holomorphic for $k = 1, 2, \cdots, J$.

Proof: Choose $\delta(\varepsilon)$ to normalize $\log h(\varepsilon)$:

$$\int_X |u_\varepsilon|^2 = \delta(\varepsilon)^2 \int_X |\log h(\varepsilon)|^2 = 1.$$

By Corollary 2.2 and Proposition 2.3, if $m(\varepsilon) = \max_X \log|h(\varepsilon)|$ we have both

$$m(\varepsilon) \leqq \varepsilon^{-1} \max_X |H_0|, \tag{4.22}$$

and

$$m(\varepsilon) \leqq C_0\Big(\delta(\varepsilon)^{-1} + \max_X |H_0|\Big).$$

It follows that

$$\begin{aligned} \int_X |u_\varepsilon|^2 \omega^n &\leqq \delta(\varepsilon)^2 m(\varepsilon)^2 \text{vol } X \\ &\leqq C_0^2 \text{ vol } X\Big(1 + \delta(\varepsilon)\max_X |H_0|\Big)^2. \end{aligned} \tag{4.23}$$

By Lemma 2.4 and this inequality we have the additional inequality (assume $m(\varepsilon)$ is large):

$$\begin{aligned} \int_X |\partial^0 u_\varepsilon|^2 \omega^n &= \delta^2(\varepsilon) \int_X |\partial^0 \log h(\varepsilon)|^2 \omega^n \\ &\leqq m(\varepsilon)^2 \delta(\varepsilon)^2 \max_X |H_0| (1 - e^{-m(\varepsilon)})^{-1} \\ &\leqq 2C_0^2\Big(1 + \delta(\varepsilon)\max_X |H_0|\Big)^2 \max_X |H_0|. \end{aligned}$$

This bounds $u_\varepsilon = \delta(\varepsilon)\log h(\varepsilon)$ in $L_1^2(L(E, E))$. Because $L_1^2(L(E, E))$ is weakly compact, and $\underline{\lim}\, \delta(\varepsilon) \to 0$, we can pick $\varepsilon_l \in [1, 0)$, $\varepsilon_l \to 0$ such that $\delta_l = \delta(\varepsilon_l) \to 0$ and $u_l = u_{\varepsilon_l} = \delta(\varepsilon_l)\log h(\varepsilon_l) \to u_\infty$ in L_1^2.

Suppose for the moment we are on an open set $X_0 \subset X$ in which the eigenspaces of u_l are of constant rank:

$$u_l = \sum_{j=1}^{J} \hat{\lambda}_{l,j} \pi_{j,l} = \sum_{j=1}^{J} \delta(\varepsilon_l) \lambda_{l,j} \pi_{j,l},$$

where $\hat{\lambda}_{l,j} = \delta(\varepsilon)\lambda_{l,j}$ are the normalized eigenvalues. By another application of the proof of (2.4),

$$\int_{X_0} |\partial^0 \lambda_{j,l}|^2 \leqq m(\varepsilon_l)^2 \max_X |H_0|.$$

Thus, for the normalized eigenvalues,

$$\int_{X_0} |\partial^0 \hat{\lambda}_{j,l}|^2 \omega^n \leqq \rho(\varepsilon)^2 m(\varepsilon_l)^2 \max_X |H_0| \to 0.$$

It follows that, as $l \to \infty$, the eigenvalues of u_l become constant on X_0. A global version on X states that all the symmetric functions of the eigenvalues of u_l approach constants as $l \to \infty$. Hence the symmetric functions of the eigenvalues of u_∞, and the eigenvalues themselves are constant a.e.

Furthermore, on $X_l \subseteq X$ we have, from (2.2) and (2.3),

$$\int_{X_l} |A_l^{jk}|^2 (e^{\lambda_{j,l} - \lambda_{k,l}} - 1)(\hat{\lambda}_{j,l} - \hat{\lambda}_{k,l}) \leqq 2 \max_X |H_0|,$$

where $|A_l^{jk}|^2 = |\langle \partial^0 V_{\pi_j} V_{\pi_k} \rangle|^2 = |\langle V_{\pi_j} \bar{\partial}, V_{\pi_k} \rangle|^2$. On the subset $X_l \subset X$, $\hat{\lambda}_{j,l} > \hat{\lambda}_{k,l}$. So, given that $\delta_l \to 0$, $\lambda_{j,l} \gg \lambda_{k,l}$. Then $(e^{\lambda_{j,l} - \lambda_{k,l}} - 1) \gg \hat{\lambda}_{j,l} - \hat{\lambda}_{k,l}$.

We conclude that

$$\int_{X_l} |\pi_{k,l}(\bar{\partial}\pi_{j,l})|^2 (\hat{\lambda}_{j,l} - \hat{\lambda}_{k,l})^2 \omega^n = \int_{X_l} |A^{j,k}|^2 (\hat{\lambda}_{j,l} - \hat{\lambda}_{k,l})^2 \omega^n \to 0,$$

in case $k > j$. Since $\hat{\lambda}_{j,l} - \hat{\lambda}_{k,l} \to a_j - a_k < 0$ we see that, for $\pi_k = \lim_{l \to \infty} \pi_{k,l}$,

$$\int_X |\pi_k(\bar{\partial}\pi_j)|^2 \omega^n = 0,$$

for $k > j$. This argument is made completely rigorous by approximating $\Sigma_{j \leqq k} \pi_k$ by the sequence $\phi_k(u_l)$, where $\phi_k\colon \mathbb{R} \to \mathbb{R}$, $\phi_k(\lambda) = 1$ for $\lambda \leqq a_k + \delta$, $\phi_k(\lambda) = 0$ for $\lambda \geqq a_{k+1} - \delta$. We approximate $\Sigma_{j > k} \pi_j$ by ${}^*q_k(u)$, where ${}^*q_k(\lambda)\phi_k(\lambda) = 0$, ${}^*q_k(\lambda) = 0$ for $\lambda \leqq a_k + \delta$, and ${}^*q_k(\lambda) = 1$ for $\lambda \geqq a_{k+1} - \delta$. Then one shows from the same inequalities that ${}^*q_k(u_l) \to {}^*q_k(u_\infty) = \Sigma_{j > k} \pi_j$ in L_1^2, $\phi_k(u_l) \to \Sigma_{j \leqq k} \pi_j$ in L_1^2, and $\int_X |({}^*q_k(u_l)\bar{\partial}({}^*q_k(u_l))|^2 \omega^n \to 0$.

5. Estimates from Yang-Mills Theory

We use the equation $H + \varepsilon \log h = 0$ as an approximation to the equation $H = 0$, a special case of the Yang-Mills equation. A bound on $\max_X |H|$ will lead us to further useful estimates.

LEMMA 5.1. *For $\varepsilon > 0$, the Hermitian-Yang-Mills tensor $H = H(h(\varepsilon))$ satisfies (in the base norm)*

$$|h^{1/2}Hh^{-1/2}| = |H| \leqq \max_X |H_0|.$$

Proof: Since $H = -\varepsilon \log h$, we see luckily that $h^{1/2}Hh^{-1/2} = H$, and a max bound on $|\varepsilon \log h|$ is required. From Corollary 2.2,

$$|\log h| \leqq \varepsilon^{-1} \max_X |H_0|$$

as required.

We will be able to see directly that, even as h_ε diverges, the associated connections converge on a dense open set as *unitary connections*, not as holomorphic connections. In one complex dimension, the compactness theorem implies that this convergence is on all of X (see [27]), in two complex dimensions it is known to be valid on the complement of a finite set of points (see [9]). We change our point of view and regard the holomorphic connection $\{\partial^0_{\bar\alpha}, \partial^0_\alpha + h^{-1}\partial^0_\alpha h\}$ as a unitary connection, and extend the results which are known in complex one-and two-dimensional spaces to arbitrary dimension.

The metric for which the new connection is unitary is h. Any $GL(r, \mathbf{C})$ gauge change v for which $h = (vv^*)^{-1}$ will transform h into the base metric. The unitary gauge group freedom lies in the arbitrariness of the choice of v. The possibilities for v are $v = uh^{-1/2}$, where u is an element of the unitary gauge group. If one were to choose a global canonical v, presumably it would be $h^{-1/2}$. Some of the estimates in Section 2 exploit this gauge change in a non-explicit fashion. Let us agree to make this gauge change, so that in the canonical unitary gauge

$$D_\alpha = \partial^0_\alpha + A_\alpha = \partial^0_\alpha + h^{-1/2}\partial^0_\alpha h^{1/2},$$

$$D_{\bar\alpha} = \partial^0_{\bar\alpha} + A_{\bar\alpha} = \partial^0_{\bar\alpha} + h^{1/2}\partial^0_{\bar\alpha} h^{-1/2},$$

$$H = h^{1/2}Hh^{-1/2}.$$

Certainly one reason for the success of our technical method is that the norm of the Hermitian-Yang-Mills tensor is not affected by this change. Of course, all our estimates are completely gauge-invariant. It is only necessary to choose the gauges consistently.

The main result we shall use from gauge field theory is an *a priori* estimate for holomorphic connections where a bound on the Hermitian-Yang-Mills tensor is known. This is a class of *a priori* estimates, which are proved *via* a scaling inequality and a geometry of X. Balls of radius σ_0 should be nearly Euclidian. The number K depends on the group, in this case on $U(r)$. Recall that the bundle being holomorphic means the $(2,0)$ and hence $(0,2)$ part of the curvature vanishes.

THEOREM 5.1 (Uhlenbeck [28]). *Let $E \to X$ be a rank r vector bundle over a compact Kähler manifold X (complex dimension n, real dimension $2n$). Given $n < p < \infty$, there exist $\sigma_0 > 0$, $K > 0$, and $C_1 < \infty$ such that if ∂^A is a unitary holomorphic connection in E and*

(a) $\max_X |H| \leqq K^2$,
(b) $\sigma^{4-2n} \int_{\delta(x,y) \leqq 4\sigma} |F|^2 \omega^n < K^2$,
(c) $\sigma \leqq \min(\sigma_0, K)$,

then

$$\int_{\delta(x,y)<2\sigma} |F|^p \omega^n < C_1 \sigma^{(2-p)n} \left(\int_{\delta(x,y) \leqq 4\sigma} |F|^2 \right)^{p/2} + \sigma^{2n} \max_X |H|^p.$$

Here we have set $H = g^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}}$ with no μI factor.

As given, this is not the estimate we need. Part of the proof of the above theorem uses the existence of "good gauges".

COROLLARY 5.1; see [27]. *Under the hypothesis of Theorem 5.1, we have the existence of a (unitary) local trivialization in $B_{2\sigma}(y)$ such that $E|B_{2\sigma}(y) = B_{2\sigma}(y) \times \mathbf{C}^N$, $D = d + A$, and*

$$\int_{\delta(x,y) \leqq 2\sigma} |\text{grad } A|^p + \sigma^{-p} \int_{\delta(x,y) \leqq 2\sigma} |A|^p \leqq C_2 \int_{\delta(x,y) \leqq 2\sigma} |F|^p.$$

How does this lead to an estimate on h? Recall our discussion on the equivalence of holomorphic and unitary approaches.

PROPOSITION 5.1. *Suppose the connection ∂^A arises from the connection ∂^0 via the metric h. Normalize the metric h so that $h \leqq I$. Then under the hypothesis of Theorem 5.1, in $B_{2\sigma}(y)$ we have $h = vv^*$, where A_α satisfies the estimates of Corollary 5.3 and*

$$\partial_\alpha^0 v = v A_\alpha.$$

Moreover,

$$\int_{\delta(x,y) \leqq \sigma} |(\partial^0)^2 h|^p \omega^n + \sigma^{-p} \int_{\delta(x,y) \leqq \sigma} |\partial^0 h|^p \omega^n$$

$$\leqq C_3 \left[\sigma^{(2-p)n} \left(\int_{\delta(x,y) \leqq 2\sigma} |F|^2 \right)^{p/2} + \sigma^{2n} \max_X |H|^p + 1 \right].$$

Proof: The terms in σ are scaling constants. We assume the ball is the unit ball and the Kähler metric nearly flat. Moreover, the original connection was C^∞ and smooth. The constant C_3 will depend on ∂^0.

Apply Theorem 5.1 and Corollary 5.1 to the connection $\partial_\alpha^0 + h^{-1/2}\,\partial_\alpha^0 h^{1/2}$, $\partial_{\bar\alpha}^0 + h^{1/2}\,\partial_{\bar\alpha}^0 h^{-1/2}$. Since an arbitrary gauge change for the unitary connection is u, we have $v^{-1}\,\partial_\alpha^0 v = A_\alpha$ for $h^{1/2}u = v$, where A_α is the difference between the original connection ∂^0 and the form for which the estimates in Corollary 5.1 are valid. However, the original connection is uniformly smooth and we have these same estimates for A_α. Elliptic estimates transfer the L_1^p-estimates on A_α in a ball to those on the L_2^p-norm of h in a ball of half the radius.

We state without proof the following convergence theorem. The argument concerning the cover of X follows that of Sedlacek [23] for real dimension four. The global gauge transformations are constructed in Donaldson [10] and Uhlenbeck [27]. The assumption of compactness made there is not necessary, as it can be replaced by a cover which is the finite union of covers made up of disjoint balls. The theorem describes the limiting behavior of our equation $H + \varepsilon \log h = 0$, or of the heat equation.

THEOREM 5.2. *Let $E \to X$ be a Hermitian vector bundle over a Kähler manifold X and D_l a sequence of holomorphic connections such that*

(a) $\int_X |F(D_l)|^2 \leqq B$,

(b) $\max_X |H(D_l)| \leqq K$.

Then there exists a subsequence $D_{l'}$ and a closed singular set of finite Hausdorf dimension $2n - 4$ such that $D_{l'}|X - \mathscr{S}|$ is gauge equivalent to a sequence of connections which converges (weakly) in $L^p_{1,\mathrm{loc}}(X - \mathscr{S})$ to an $L^p_{1,\mathrm{loc}}$ holomorphic connection on $X - \mathscr{S}$.

COROLLARY 5.2. *Let E be the Hermitian vector bundle associated to E and the base metric, and B be the holomorphic, unitary connections in E which are* $\mathrm{GL}(r, \mathbf{C})$ *equivalent to those constructed from the metric solutions to $H + \varepsilon \log h =$ 0. Then there exists a subsequence $D_{l'}$ and a singular set of finite Hausdorf dimension $2n - 4$ such that $D_{l'}$ are gauge equivalent to a sequence which converges in L_1^p on $X - \mathscr{S}$. Moreover, $\pi = I - \lim_{s\to 0}\lim_{l\to\infty} h^s$ is meromorphic on $X - \mathscr{S}$.*

Proof: By Lemma 5.1, $\max_X |H|$ is bounded, and by Proposition 3.3, $\int_X |F|^2$ is bounded, so we can apply the theorem. The regularity of A_α is sufficient to show that the cokernel is meromorphic.

THEOREM 5.3. *The projection π constructed in Corollary 5.2 represents a holomorphic subsheaf of E.*

Proof: We need only show that the singularities of real codimension $2n - 4$ are actually the singularities of a meromorphic function (see [24]) and that a meromorphic function from X into a Grassmannian pulls back the canonical vector bundle to a coherent, reflexive sheaf. This is a much simpler version of the discussion in Section 7 and we refer the reader to that.

As a biproduct of our theory, we have a result concerning the compactness of the space of Hermitian-Yang-Mills connections in $E \to X$. Since these parameterize the stable holomorphic bundles $\mathscr{E}$ which are topologically E, this result should relate other methods of compactifying the space of stable bundles. We conjecture that the limit object has a natural structure as a sheaf, although even for complex surfaces the actual construction of the sheaf is unclear.

THEOREM 5.4. *Let D_l be a sequence of holomorphic Hermitian-Yang-Mills connections in a Hermitian vector bundle E over a compact Kähler manifold X. Then there exists a gauge equivalent subsequence $D_{l'}$, and a closed set $\mathscr{S}$ of (real) Hausdorf dimension $2n-4$ such that $D_{l'}|X-\mathscr{S}| \to D_\infty$. Here D_∞ is a holomorphic Hermitian-Yang-Mills connection in $E|X-\mathscr{S}|$.*

For complex surfaces, the removable singularities theorem shows that D_∞ completes to a holomorphic Hermitian-Yang-Mills connection in E_∞. However, E_∞ is topologically different from E if $\mathscr{S} \neq \varnothing$ in that at every point in $\mathscr{S}$ at which the convergence fails, of necessity integral multiples of C_2 are absorbed, and the second Chern class of the bundle E_∞ is smaller than that of E.

6. Regularity of Weakly Holomorphic Maps

The purpose of this section is to prove that a weakly holomorphic map into an algebraic manifold is meromorphic. This will be used in the next section to show the regularity of the subsheaf $\mathscr{F}$. To be precise, let us first define a weakly holomorphic map from the ball $B \subset \mathbb{C}^n$ into an algebraic manifold M. We assume that M is isometrically embedded in $\mathbb{CP}^K$ which is embedded in some Euclidian space $\mathbb{R}^N$. The map $F: B \to M \subset \mathbb{R}^N$ is then a vector-valued map and it makes sense to say that it is L_1^2. By definition, this means that there is a sequence of smooth maps F_i from B into $\mathbb{R}^N$ so that both F_i and dF_i converge in $L^2(B)$ and that $\lim_{i\to\infty} F_i = F$. As usual, we define dF to be $\lim_{i\to\infty} dF_i$.

When $n=1$, we define F to be weakly holomorphic if it is L_1^2 and, for almost every point x in B, dF sends the holomorphic tangent space of $\mathbb{C}T_x(B)$ into the holomorphic tangent space at $F(x)$ of $\mathbb{C}T_{F(x)}(M)$. Note that this makes sense because for almost every point x in B, dF sends the tangent space of B to the tangent space of M. (In fact, in this case, F_i can be assumed to have the image in M (see [21])).

When $n > 1$, we call F weakly holomorphic if for any linear holomorphic coordinate system $\{z_1, \cdots, z_n\}$ on B and for almost every point $\{z_2, \cdots, z_n\}$, the restrictions of F to each complex line in the z_1 variable is weakly holomorphic.

THEOREM 6.1. *Any weakly holomorphic map into an algebraic manifold is meromorphic.*

Proof: First of all, we prove the theorem when $n=1$.[1] It is clear that we have only to prove the continuity of F.

[1] M. Gruter, *Regularity of weak H-surface*, J. Reine Angew. Math., 329, 1981, pp. 1–15.

Let $x \in B$ be any point. Let $B_x(r)$ be the balls of radius r around x and $E_x(r)$ the energy of F over $B_x(r)$. Then we want to prove that, for some $\alpha > 0$, $E_x(r)r^{-\alpha}$ is bounded as x varies in a compact set of B. The continuity of F will then follow from Morrey's lemma; see [16].

Clearly, we can assume that r is small and $B_x(r) \subset B$. As the total energy is bounded, it suffices to prove the differential inequality

$$\frac{d\left(r^{-1}E_r\right)}{dr} = r^{-1}\frac{dE_r}{dr} - r^{-2}E_r \geqq 0.$$

Therefore, $E_x(r) \leqq r\,dE/dr$. If the image of the circle of radius r is large, then $\int_0^{2\pi}|dF/d\theta|^2\,d\theta \leqq r\,dE/dr$ is large and the inequality is obvious. Hence we can assume that the length of these curves is uniformly small.

When the length of the image of the circle of radius r is small, it lies in a coordinate chart of M. One can solve the Plateau problem for this curve within this coordinate chart. By the isoperimetric inequality for minimal surfaces, we know that the area of this minimal surface is dominated by the square of the length of the curve. Hence the curve bounds the images of two different disks. In particular, we have a map of the sphere into M. (The map F on $B_x(r)$ is not *a priori* continuous. We simply approximate F in L_1^2-norm by a smooth map into M. This is possible because $n = 1$ (see [22]).) We claim that this map is homotopically trivial.

In fact, by using the uniformization theorem, we may assume that the above map is conformal. In this case, area is the same as energy. Hence the total energy of this map is not greater than $E_x(r)$ plus some multiple of the square of the length of the image circle of radius r. Hence by choosing r small enough, the total energy of the map from the sphere is less than some fixed constant. As was proved in Sacks-Uhlenbeck [20], this constant can be chosen so that the map must be homotopically trivial.

Let Ω be the Kähler form on M. Then, using the Wirtinger inequality and $d\Omega = 0$, we can prove that $\int_{B_x(r)} F^*\Omega$ is not greater than the area of the minimal surface constructed above. In particular, the isoperimetric inequality shows that $\int_{B_x(r)} F^*\Omega$ is not greater than some constant times the square of the length of $F(\partial B_x(r))$.

To prove our differential inequality, it suffices to prove that $\int_{B_x(r)} \tilde{F}^*\Omega$ is $E_x(r)$. Actually, at this moment, $\tilde{F}$ is only a smooth map which approximates the original F in the L_1^2-map. In general, for smooth maps $\tilde{F}$,

$$\int_{B_x(r)} \tilde{F}^*\Omega = \int_{B_x(r)} |\partial \tilde{F}|^2 - \int_{B_x(r)} |\bar{\partial}\tilde{F}|^2.$$

We claim that when $\tilde{F}$ is close enough to our map F in L_1^2-norm, $\int_{B_x(r)} \tilde{F}^*\Omega$ is close to $E_x(r)$.

In fact, given any small $\varepsilon > 0$, we can assume that $\tilde{F}(x)$ is ε-close to $F(x)$ for x outside a set where the energy of $\tilde{F}$ is smaller than ε. For any two points y_1

and y_2 in M which has distance less than ε, we can identify the tangent space of M at y_1 and y_2 by parallel transportation in M along the shortest geodesic joining y_1 and y_2. For each point y in M, let P_y be the projection from the complexified tangent space of $\mathbb{R}^N$ at y to the holomorphic tangent space of M at y. In this terminology, $|(\bar\partial \tilde F)(x)|$ is equal to $|P_{\tilde F(x)} \tilde F_*(\partial/\partial \bar z)|$. By assumption, this is close to $|P_{\tilde F(x)} F_*(\partial/\partial \bar z)|$ in L^2-topology.

Outside a set where the energy of F is small, the L^2-norm of $|P_{F(x)} \tilde F_*(\partial/\partial \bar z)|$ is not greater than the L^2-norm $|P_{F(x)} \tilde F_*(\partial/\partial \bar z)|$ plus a small multiple of the L^2-norm of $|F_*(\partial/\partial \bar z)|$. Therefore, the L^2-norm of $|\bar\partial \tilde F|$ is not greater than a small constant plus the L^2-norm of $|\bar\partial F|$. Since $\bar\partial F = 0$ almost everywhere, we conclude that the L^2-norm of $|\bar\partial \tilde F|$ is small.

A similar reasoning shows that the L^2-norm of $|\partial \tilde F|$ is close to the L^2-norm of $|\partial F|$. Hence $\int \tilde F^* \Omega$ is close to the energy of F and we have proved our claim.

This claim implies the continuity of F and proves the theorem for $n = 1$. For $n > 1$, we proceed as follows.

Since an algebraic manifold is a submanifold of $\mathbb{CP}^n$, we assume that $M = \mathbb{CP}^n$. By writing $\mathbb{CP}^n$ as $\mathbb{C}^n$ union the plane at infinity, we can write $F = (f_1, \cdots, f_n)$, where we allow the f_i to have values at infinity. Hence Theorem 6.1 can be reduced to the following theorem which is of independent interest. (Clearly the theorem can be generalized to $\mathbb{C}^n$ with $n > 2$.)

THEOREM 6.2. *Let f be a measurable function defined on the unit polydisk $D \times D$ in $\mathbb{C}^2$. Suppose that, for some set E of measure zero in $D \times D$, the restriction of f to $D \times D \setminus E$ satisfies the following property. For almost every z, $f(z, w)$ can be extended to be a meromorphic function of w. Similarly, for almost every w, $f(z, w)$ can be extended to be a meromorphic function of z. Then f is equal to a meromorphic function almost everywhere.*

First of all, we can assume that, for almost every z, $f(z, w)$ is a meromorphic function of w.

LEMMA 6.1. *Let $E \subset D \times D$ be a subset with measure zero. Then there exists a sequence $\{z_i\} \subset D$ and a subset G of D such that $\mu(G) = \mu(D)$, where μ is the standard measure on D, $(\{z_i\} \times G) \cap E = \varnothing$ and $z_i \to \frac{1}{2} \in D$ as $i \to +\infty$.*

Proof: Since $\mu(E) = 0$, one can find a full measure subset F of D such that, for all $z \subset F$, $\mu((z \times D) \cap E) = 0$.

We choose a sequence $\{z_i\} \subset F$ such that $z_i \to \frac{1}{2}$ as $i \to +\infty$. If we set $G = D \setminus (\bigcup_{i=1}^{\infty} (z_i \times D) \cap E)$, then $\mu(G) = \mu(D)$ and $(z_i \times G) \cap E = \varnothing$.

By assumption, there is, in case $n = 1$, an $\tilde f$ on $D \times D$ such that, for almost all $w \in D$, $\tilde f(\cdot, w)$ is a meromorphic function of z and $E = \{x \in B | f(x) \neq \tilde f(x)\}$ has measure zero.

Now we choose z_i as in Lemma 6.1; then, for almost all $w \in D$ and for all i, $\tilde f(z_i, w) = f(z_i, w)$.

Taking

$$f_i(z,w) = \sum_{j=1}^{i} f(z_i, w) \prod_{\substack{k=1 \\ k\neq j}} (z - z_k)(z_j - z_k)^{-1},$$

we obtain a sequence of meromorphic functions $f_i(z,w)$, such that $f_i(z_j, w) = f(z_j, w)$ for $1 \leqq j \leqq i$. Now we claim that we can assume that both f_i and $1/f_i$ are in L^p for all $p < 2$, and the L^p-norms are uniformly bounded.

To see this, we notice that for any measurable function f, the integral $\int_{|a|<1} |f - a|^{-p}\, da\, d\bar{a}$ is bounded independent of f. Therefore, we can apply Fubini's theorem to conclude that the double integral

$$\int_{|a|<1}\left(\int_{|z|<1}\int_{|w|<1} |f_i(z,w) - a|^{-p}\, dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}\right) da \wedge d\bar{a}$$

is bounded uniformly. Then we can find a_i and b_i so that $|a_i| \leqq \frac{1}{4}$, $|b_i - \frac{3}{4}| \leqq \frac{1}{4}$, and both $(f_i - a_i)^{-1}$ and $(f_i - b_i)^{-1}$ are L^p integrable for $1 \leqq p < 2$, the L^p-norms are bounded independently of i. Let $g_i = 1 + (a_i - b_i)(f_i - a_i)^{-1}$, then the L^p-norms of both g_i and g_i^{-1} have a uniform upper bound. It is clear that $\{g_i\}$ satisfies all the hypothesis for $\{f_i\}$. Therefore we can assume that both g_i and g_i^{-1} are L^p-integrable for all $p < 2$ and the L^p-norms are uniformly bounded.

LEMMA 6.2. *Let ϕ be a meromorphic function defined on the unit disk $D = \{|z| < 1\}$. Then there exists a constant C independent of ϕ, such that*

$$\int_{|z|<3/4} \frac{\partial\phi \wedge \overline{\partial\phi}}{\left(1 + |\phi|^2\right)^2} \leqq C\left(\int_{|z|<1} \left[\log^2(1 + |\phi|^2) + \log^2(1 + |\phi^{-1}|^2)\right] + 1\right).$$

Proof: Since

$$\frac{\partial^2}{\partial z\, \partial \bar{z}} \log\left(1 + |\phi|^2\right) - \frac{\partial\phi \wedge \overline{\partial\phi}}{\left(1 + |\phi|^2\right)^2}$$

is a linear combination of the δ functions at the poles of ϕ,

$$\int_{|z|<1} p^2\left[\log^{-1}(1 + |\phi|^2)\frac{\partial^2}{\partial z\, \partial\bar{z}}\log\left(1 + |\phi|^2\right) - \log^{-1}(1 + |\phi|^2)\frac{\partial\phi \wedge \overline{\partial\phi}}{\left(1 + \phi|^2\right)^2}\right] = 0,$$

where p is a smooth function with compact support in $|z| < 1$, and equal to 1 on $|z| < \frac{3}{4}$. In the following, we denote by C a constant independent of ϕ.

Integrating by part, we obtain

$$-2\int_{|z|<1} p \log^{-1}(1+|\phi|^2)\, \partial p \wedge \bar{\partial} \log(1+|\phi|^2)$$

$$+\int_{|z|<1} p^2 \log^{-2}(1+|\phi|^2) \frac{|\phi|^2\, \partial\phi \wedge \overline{\partial\phi}}{(1+|\phi|^2)^2}$$

$$-\int_{|z|<1} p^2 \log^{-1}(1+|\phi|^2) \frac{\partial\phi \wedge \overline{\partial\phi}}{(1+|\phi|^2)^2} = 0.$$

Clearly the same equations hold if we replace ϕ by ϕ^{-1}. Summing the two equations, we obtain

$$-2\int_{|z|<1} p \log^{-1}(1+|\phi|^2) \frac{\phi\, \partial p \wedge \overline{\partial\phi}}{1+|\phi|^2} + 2\int_{|z|<1} p \log^{-1}(1+|\phi^{-1}|^2) \frac{\bar{\phi}^{-1}\, \partial p \wedge \overline{\partial\phi}}{1+|\phi|^2}$$

$$+\int_{|z|<1} p^2 \big[|\phi|^2 \log^{-2}(1+|\phi|^2) - \log^{-1}(1+|\phi|^2)$$

$$+|\phi^{-1}|^2 \log^{-2}(1+|\phi^{-1}|^2) - \log^{-1}(1+|\phi^{-1}|^2)\big] \frac{\partial\phi \wedge \overline{\partial\phi}}{(1+|\phi|^2)^2} = 0.$$

Let $h(t) = \frac{1}{4} t \log^{-2}(1+t)$ when $t \geqq 1$ and $h(t) = \frac{1}{4} t^{-1} \log^{-2}(1+t^{-1})$ when $t \leqq 1$. Then

$$|\phi|^2 \log^{-2}(1+\phi|^2) - \log^{-1}(1+|\phi|^2) + |\phi^{-1}|^2 \log^{-2}(1+|\phi^{-1}|^2)$$

$$-\log^{-1}(1+|\phi^{-1}|^2) \geqq h(|\phi|^2).$$

Therefore, applying the Schwartz inequality to the previous equation, we find

$$\int_{|z|<1} p^2 h(|\phi|^2) \frac{\partial\phi \wedge \overline{\partial\phi}}{(1+|\phi|^2)^2}$$

$$\leqq 4\left(\int_{|z|<1} p^2 h(|\phi|^2) \frac{\partial\phi \wedge \overline{\partial\phi}}{(1+|\phi|^2)^2}\right)^{1/2}$$

$$\times\bigg(\int_{|z|<1} |\nabla p|^2 h(|\phi|^2)^{-1} \big(|\phi|^2 \log^{-2}(1+|\phi|^2)$$

$$+|\phi^{-1}|^2 \log^{-2}(1+|\phi^{-1}|^2)\big)\bigg)^{1/2}.$$

By direct computation we obtain

$$h(|\phi|^2)^{-1}\left(|\phi|^2\log^{-2}(1+|\phi|^2)+|\phi^{-1}|^2\log^{-2}(1+|\phi^{-1}|^2)\right)$$
$$\leqq C\left(1+\log^2(1+|\phi|^2)+\log^2(1+|\phi^{-1}|^2)\right).$$

Since $h(|\phi|^2)\geqq C>0$, we conclude that

$$\int_{|z|\leq 3/4}\frac{\partial\phi\wedge\overline{\partial\phi}}{\left(1+|\phi|^2\right)^2}\leqq C\int_{|z|<1}\left[1+\log^2(1+|\phi|^2)+\log^2(1+|\phi^{-1}|^2)\right].$$

COROLLARY 6.1. *$\iint_{|z|\leqq 3/4,\,|w|\leqq 3/4}|\nabla f_i|^2/(1+|f_i|^2)^2\leqq C$, where C is independent of f_i.*

Proof: Applying Lemma 6.2 to $f_i(z,\cdot)$ for all $z\in D$, we have

$$\int_{|w|\leqq 3/4}\frac{\partial_w f_i\wedge\overline{\partial_w f_i}}{\left(1+|f_i|^2\right)^2}\leqq C\int_{|w|<1}\left(1+\log^2(1+|f_i|^2)+\log^2(1+|f_i^{-1}|^2)\right).$$

Therefore,

$$\iint_{\substack{|z|<1\\|w|\leqq 3/4}}\frac{|\nabla_w f_i|^2}{\left(1+|f_i|^2\right)^2}\leqq C\iint_{\substack{|z|<1\\|w|<1}}\left(1+\log^2(1+|f_i|^2)+\log^2(1+|f_i^{-1}|^2)\right).$$

Similarly, if we replace f by $\tilde{f}$, we get

$$\iint_{\substack{|z|\leqq 3/4\\|w|<1}}\frac{|\nabla_z\tilde{f}_i|^2}{\left(1+|\tilde{f}_i|^2\right)^2}\leqq C\iint_{\substack{|z|<1\\|w|<1}}\left(1+\log^2(1+|\tilde{f}_i|^2)+\log^2(1+|\tilde{f}_i^{-1}|^2)\right)$$

but $f_i=\tilde{f}_i$ a.e., and f_i, f_i^{-1} are uniformly L^p-bounded for $1\leqq p<2$,

$$\iint_{\substack{|z|\leqq 3/4\\|w|\leqq 3/4}}\frac{|\nabla f_i|^2}{\left(1+|f_i|^2\right)^2}\leqq C.$$

Now we consider the graphs of f_i as subsets of $D\times D\times\mathbb{CP}^1$; then the above corollary implies that, over each compact subset of $D\times D$, the volumes of the graphs are uniformly bounded, since the volume form of the graph of f_i over $D\times D$ is dominated by the Euclidean volume form of $D\times D$ plus

$$\frac{\partial f_i\wedge\overline{\partial f_i}}{\left(1+|f_i|^2\right)^2}\left(dz\wedge d\bar{z}+dw\wedge\overline{dw}\right).$$

Therefore, by Bishop's theorem [5], the graphs of $|f_i|$ converge to an analytic subset A of $D \times D \times \mathbb{CP}^1$. Furthermore, for each $(z, w) \in D \times D$, the limit set of $\{f_i(z, w)\}$ is contained in $A \cap ((z, w) \times \mathbb{CP}^1)$. We choose a meromorphic function h on $D \times D \times \mathbb{CP}^1$ so that for each point $x \in (z, w) \times \mathbb{CP}^1$ which is not in the limit set h is holomorphic on $(z, w) \in D \times D$ and $h(A) = 0$, $h(x) \neq 0$. Define $h_n(y) = h(y) - h(f_n(z', w'))$, for $y \in (z', w') \times \mathbb{CP}^1$; then $h_n = 0$ on the graph of f_n. Now $h_n \to \tilde{h}$, $\tilde{h}$ vanishes on A, but $\tilde{h}(x) \neq 0$, so $x \notin A$, i.e., the limit set of $\{f_i(z, w)\}$ is just $A \cap ((z, w) \times \mathbb{CP}^1)$. Since the projection of $D \times D \times \mathbb{CP}^1$ onto $D \times D$ has degree one on the graph of each f_i, there exists an analytic subset S of codimension one in $D \times D$ such that $A \setminus (S \times \mathbb{CP}^1)$ is a graph over $D \times D \setminus S$. For each j, $f_i(z_j, w) = f(z_j, w)$ for $i \geqq j$. The limit set of $\{f_i(z_j, w)\}$ consists of only one point $(z_j, w; f(z_j, w))$, i.e., $(z_j, w) \in D^0 \times D^0 \setminus S$.

Hence, we conclude that there exists a meromorphic function g in $D \times D$, such that $g(z_j, w) = f(z_j, w)$ for $j \geqq 1$. Since, for almost all $w \in G \subset D$, $\tilde{f}(z, w)$ is meromorphic on z, and is equal to $g(z, w)$ at infinite number of points $\{z_i\}$, $g(z, w) \equiv \tilde{f}(z, w)$ for such $w \in G$, i.e., $g(z, w) = \tilde{f}(z, w)$ a.e.

It follows that f is equal to a meromorphic function almost everywhere. Theorem 6.2 is proved.

Remark. As was noted by B. Shiffman, a corollary of Theorem 6.2 is that every closed locally rectifiable (1, 1) current is given by a holomorphic chain. This statement was proved by R. Harvey and B. Shiffman in [12] with the extra assumption on the support of the current. More recently, Shiffman informed us that he has a different method to demonstrate the statement.

7. Regularity of Weakly Holomorphic Subbundles

In this section, we describe another approach to prove the regularity of π without using the estimate from Yang-Mills theory. In fact, we shall prove that the weakly holomorphic subbundle defined in Section 4 is given by a holomorphic subsheaf.

Given a holomorphic vector bundle E over a complex manifold M, we can find an open cover $\{O_i\}$ of M so that $E|O_i$ is biholomorphic to $O_i \times \mathbb{C}^r$ and the transition transformations are given by holomorphic maps from $O_i \cap O_j$ to $\mathrm{GL}(r, \mathbb{C})$. A holomorphic subbundle of rank k can be given as follows. For each i, it is given by a holomorphic map f_i from O_i to $G(r, k)$, the Grassmanian of k planes in $\mathbb{C}^r$. The map f_i is equal to f_j up to the automorphism of $G(r, k)$ induced from the transition transformations of the bundle E.

If f_i is only rational, i.e., f_i is defined outside a subvariety of codimension at least two in U_i, then the set $\{f_i\}$ defines a coherent subsheaf of E. It can be seen in the following way. Let $U(k)$ be the universal bundle over $G(r, k)$, F_i be the closure of the graph of f_i. Let π_1^i, π_2^i be the projections to the first and the second factor, respectively. Then $\pi_2^{i*}U(k)$ is a k-bundle on F_i, by the Gauert direct image theorem, $\pi_{i*}^{8}(\pi_2^{i*}U(k))$ is a coherent sheaf on O_i, induced by f_i. Obviously, $\pi_{1*}^{i}(\pi_2^{i*}U(k))$ and $\pi_{1*}^{\kappa}(\pi_{\psi}^{\kappa*}U(k))$ are patched up on $O_i \cap O_j$ by the automorphism of $G(r, k)$, since f_i and f_j are.

A weakly holomorphic subbundle is given by a projection map π. The rank of the projection is given by $\operatorname{tr}\pi$ which has to be an integer. Since $\operatorname{tr}\pi$ is L_1^2, it is straightforward to show that the integer is in fact a constant almost everywhere. Let $\operatorname{tr}\pi = k$ be this constant. Then, in each coordinate chart O_i, π induces a mapping $\hat{\pi}_i$ from a set of O_i with full measure into $G(r, k)$. We claim that the $\hat{\pi}_i$ can be extended to be a rational mapping from O_i to $G(r, k)$. Without losing generality, we may assume that $n = 2$, $O = O_i = D \times D$, $\hat{\pi} = \hat{\pi}_i$, where D is the unit disk in C^1. As π is L_1^2, for almost every $z \in D$ (respectively $w \in D$), the restriction of π to $\{z\} \times D$ (respectively to $D \times \{w\}$) is L_1^2. Consider π to be a mapping from $D \times D$ to the submanifold V_k of End E consisting of all elements with rank equal k. The well-known approximation theorem (see [22]) tells us that, if $\pi_z = \pi\{z\} \times D$ is L_1^2, π_z can be approximated by L_1^2-bounded smooth mappings F_i from $\{z\} \times D$ into V_k. As discussed in Section 6, we see that $\lim_{i\to\infty}\int_{O\times\Delta}|(I - F_i)\bar{\partial}F_i|^2 = 0$. Define $\hat{F}_i\{z\} \times D \to G(r, k)$ by $\hat{F}_i(z, w) =$ the image $F_i(z, w)(C^r) \in G(r, k)$. We claim that the limit of $\hat{F}_i$ gives a rational extension of $\hat{\pi}_z = \hat{\pi}|_{z\times\Delta}$.

First of all, let us prove that the $\hat{F}_i$ are L_1^2, and their L_1^2-norms are uniformly dominated by those of π_i and so are uniformly bounded. We use the Plücker coordinate for $G(r, k)$. Let $e_1, e_2, \cdots, e_r$ be any unitary frame field which spans $F_i(C^r)$ at any given point $(z, w) \in z \times D$, so that $F_i(e_j) = e_j$ for $j \leqq k$. Since F_i is continuous, $F_i(e_1) \wedge \cdots \wedge F_i(e_k)$ defines the plane in $G(r, k)$ near (z, w) and, at (z, w), $|\nabla[F_i(e_1) \wedge \cdots \wedge F_i(e_k)]|$ is dominated by $|dF_i|$ for any subset $\{i_1, \cdots, i_k\}$ of $\{1, 2, \cdots, n\}$. This implies that the L_1^2-norms of $\hat{F}_i$ are dominated by those of F_i.

Secondly, we prove that $\lim_{i\to+\infty}\int_{O\times\Delta}|\bar{\partial}\hat{F}_i|^2\,dw = 0$. This follows from the definition of $\hat{F}_i$ and

$$\lim_{i\to+\infty}\int_{O\times\Delta}|(I - F_i)\bar{\partial}F_i|^2\,dw = 0.$$

In fact, let $e_1, \cdots, e_r$ be a holomorphic frame of C^r, so that, at a given point (z, w), $F_i(e_j) = e_j$ for $1 \leqq j \leqq k$, and $e_1, \cdots, e_r$ are orthogonal. One checks that

$$\begin{aligned}
&\bar{\partial}[F_i(e_1) \wedge \cdots \wedge F_i(e_k)]\\
&\quad= \sum_{m=1}^{k}\langle\bar{\partial}F_i(e_j), e_j\rangle F_i(e_1) \wedge \cdots \wedge F_i(e_k)\\
&\qquad+ \sum_{j=1}^{k}(-1)^{j-1}F_i(e_1) \wedge \cdots \wedge (I - F_i)\bar{\partial}F_i(e_j) \wedge \cdots \wedge F_i(e_k)\\
&\quad= \operatorname{tr}(F_i\bar{\partial}F_i)\cdot F_i(e_1) \wedge \cdots \wedge F_i(e_k)\\
&\qquad+ \sum_{j=1}^{k}(-1)^{j-1}F_i(e_1) \wedge \cdots \wedge (I - F_i)\bar{\partial}F_i(e_j) \wedge \cdots \wedge F_i(e_k).
\end{aligned}$$

Therefore, $|\bar{\partial}\hat{F}_i|$ is dominated by $|(I - F_i)\bar{\partial}F_i|$ and $|\mathrm{tr}(F_i\bar{\partial}F_i)|$, $\mathrm{tr}(F_i\bar{\partial}F_i)$ converges to $\mathrm{tr}(\pi_z\bar{\partial}\pi_z)$ in L^2 and $\mathrm{tr}(\pi_z\bar{\partial}\pi_z) = \mathrm{tr}(\bar{\partial}\pi_z)$, $\bar{\partial}\,\mathrm{tr}\,\pi_z = 0$ since $\mathrm{tr}(\pi_z)$ is constant almost everywhere. We conclude that $\int_{Z|\Delta}|\mathrm{tr}(F_i\bar{\partial}F_i)|^2\,dw \to 0$ as $i \to +\infty$. On the other hand, $\int_{z\times\Delta}|(I - F_i)\bar{\partial}F_i|^2\,dw \to 0$ as $i \to +\infty$, hence $\lim_{i\to+\infty}\int_{O\times\Delta}|\bar{\partial}\hat{F}_i|^2\,dw = 0$.

Summing the above, we see that the limit of $\hat{F}_i$ gives an extension of $\hat{\pi}_z = \hat{\pi}|_{z\times\Delta}$, which is weakly holomorphic. By the regularity theorem of Section 6, it follows that, for almost every $z \in D$, $\hat{\pi}|_{O\times\Delta}$ can be extended to be a holomorphic mapping from $\{z\} \times D$ into $G(r, k)$. Similarly, for almost every $w \in D$, $\hat{\pi}|_{D\times\psi}$ can be extended to be a holomorphic mapping from $D \times \{w\}$ into $G(r, k)$.

Now our claim follows from the regularity theorem of Section 6 for $n = 2$, so does the main theorem.

8. Applications

Our main application is similar to a major application of the solution of the Calabi conjecture: inequalities between Chern numbers of the tangent bundle [31]. The same argument gives the following:

THEOREM 8.1. *Let E be a stable holomorphic bundle over a compact n-dimensional Kähler manifold X with respect to the Kähler class ω. Let $r = \mathrm{rank}\ E$. Then $2rC_2(E) \cup \omega^{n-2} \geqq (r + 1)C_1^2 \cup \omega^{n-2}$ and the equality holds if and only if the pull-back of E to the universal cover of X is a direct sum of Hermitian holomorphic line bundles where the curvature forms of these line bundles are equal to each other.*

COROLLARY 8.1. *Let E be a stable holomorphic bundle over a compact Kähler manifold X. Then E is obtained from unitary representation of the fundamental group of X if and only if $C_1^2 \cup \omega^{n-2} = 0$ and $C_2 \cup \omega^{n-2} = 0$.*

Using the connection constructed in this paper, Cho, in his Harvard thesis [7], computed the curvature of the Kähler metric on the moduli space of stable vector bundles. He concluded that the holomorphic sectional curvature must be positive if X is a curve. In particular, if the moduli space is non-singular and compact, it must be simply connected.

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